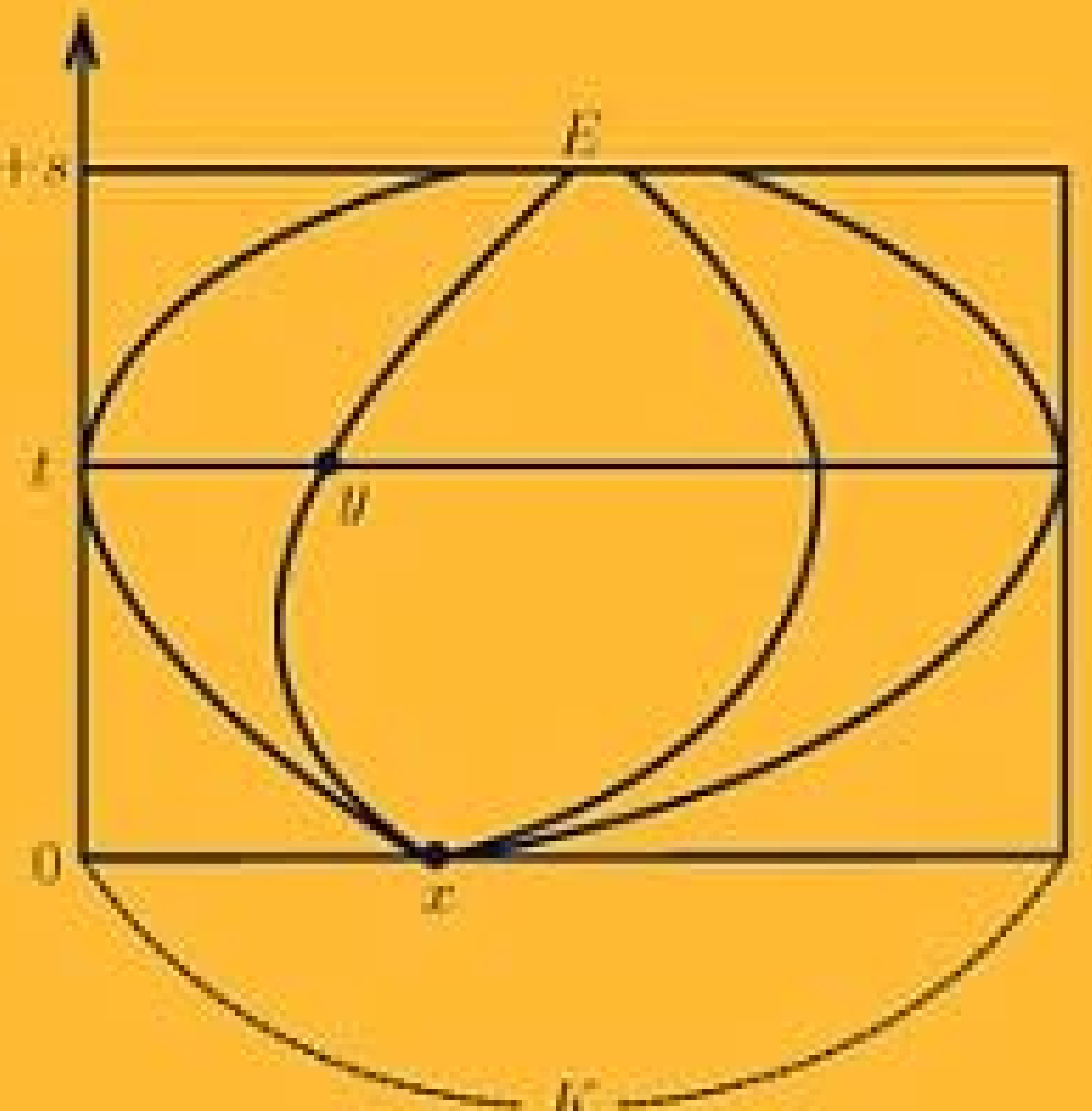
Lecture Notes in Mathematics

1499

Kazuaki Taira

Boundary Value Problems and Markov Processes

2nd Edition



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To the memory of
Professor Kiyosi Itô
(1915–2008)

Preface to the Second Edition

This monograph is an expanded and revised version of a set of lecture notes for the graduate courses given by the author both at Hiroshima University (1995–1997) and at the University of Tsukuba (1998–2000) which were addressed to the advanced undergraduates and beginning-graduate students with interest in functional analysis, partial differential equations and probability.

The first edition of this monograph, which was based on the lecture notes given at the University of Tsukuba (1988–1990), was published in 1991. This edition was found useful by a number of people, but it went out of print after a few years.

This second edition has been revised to streamline some of the analysis and to give better coverage of important examples and applications. The errors in the first printing are corrected thanks to kind remarks of many friends. In order to make the monograph more up-to-date, additional references have been included in the bibliography.

This second edition may be considered as a short introduction to the more advanced book "*Semigroups, boundary value problems and Markov processes*" which was published in the Springer Monographs in Mathematics series in 2004. For graduate students working in functional analysis, partial differential equations and probability, it may serve as an effective introduction to these three interrelated fields of analysis. For graduate students about to major in the subject and mathematicians in the field looking for a coherent overview, it will provide a method for the analysis of elliptic boundary value problems in the framework of L^p spaces.

My special thanks go to the editorial staffs of Springer-Verlag for their unfailing helpfulness and cooperation during the production of this second edition.

This research was partially supported by Grant-in-Aid for General Scientific Research (No. 19540162), Ministry of Education, Culture, Sports, Science and Technology, Japan.

Last but not least, I owe a great debt of gratitude to my family who gave me moral support during the preparation of this book.

Tsukuba, Kazuaki Taira
March 2009

Preface to the First Edition

This monograph is devoted to the functional analytic approach to a class of *degenerate* boundary value problems for second-order elliptic differential operators which includes as particular cases the Dirichlet and Neumann problems. We prove that this class of boundary value problems provides a new example of *analytic semigroups* both in the L^p topology and in the topology of uniform convergence. As an application, we show that there exists a strong *Markov process* corresponding to such a diffusion phenomenon that either absorption or reflection phenomenon occurs at each point of the boundary. Furthermore, we study a class of initial-boundary value problems for *semilinear* parabolic differential equations.

This monograph is an expanded version of a set of lecture notes for the graduate courses given by the author at the University of Tsukuba between 1988 and 1990. We confined ourselves to the simple boundary condition. This makes it possible to develop our basic machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly. I hope that this monograph will lead to a better insight into the study of three interrelated subjects: elliptic boundary value problems, analytic semigroups and Markov processes. For additional information on many of the topics discussed here, I would like to call attention to my previous book *Diffusion Processes and Partial Differential Equations*, Academic Press, 1988.

I would like to express my hearty thanks to many colleagues and graduate students in my courses, whose helpful criticisms of my lectures resulted in a number of improvements. The manuscript is typeset in a camera-ready form using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

Tsukuba, April 1991 — Kazuaki Taira

Contents

1

Introduction and Main Results

In this introductory chapter, our problems and results are stated in such a fashion that a broad spectrum of readers could understand, and also described how these problems can be solved, using the mathematics presented in Chapters 2 through 4.

In 1828, the English botanist R. Brown observed that pollen grains suspended in water move chaotically, incessantly changing their direction of motion. The physical explanation of this phenomenon is that a single grain suffers innumerable collisions with the randomly moving molecules of the surrounding water. A mathematical theory for Brownian motion was put forward by A. Einstein in 1905 ([Ei]). Einstein derived an accurate method of measuring Avogadro's number by observing particles undergoing Brownian motion. Einstein's theory was experimentally tested by J. Perrin between 1906 and 1909.

Brownian motion was put on a firm mathematical foundation for the first time by N. Wiener in 1923 ([Wi]). Wiener characterized the "starting afresh" property of Brownian motion that if a Brownian particle reaches a position, then it behaves subsequently as though that position had been its initial position.

Markov processes are an abstraction of the idea of Brownian motion. In the first works devoted to Markov processes, the most fundamental was A. N. Kolmogorov's work in 1931 ([Ko]) where the general concept of a Markov transition function was introduced for the first time and an analytic method of describing Markov transition functions was proposed. From the viewpoint of analysis, the transition function is something more convenient than the Markov process itself. In fact, it can be shown that the transition functions of Markov processes generate solutions of certain parabolic partial differential equations such as the classical diffusion equation; and, conversely, these differential equations can be used to construct and study the transition functions and the Markov processes themselves.

In the 1950s, the theory of Markov processes entered a new period of intensive development. We can associate with each transition function in a natural

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way a family of bounded linear operators acting on the space of continuous functions on the state space, and the Markov property implies that this family forms a semigroup. The Hille–Yosida theory of semigroups in functional analysis made possible further progress in the study of Markov processes. The semigroup approach to Markov processes can be traced back to the pioneering work of Feller in early 1950s ([Fe1], [Fe2]).

Now let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$A = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a second-order, *elliptic* differential operator with real smooth coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$ and $1 \leq i, j \leq N$.
(2) There exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{D} \times \mathbf{R}^N.$$

(3) $c(x) \leq 0$ on $\overline{D}$.

We consider the following boundary value problem with spectral parameter: Given functions $f(x)$ and $\varphi(x')$ defined in D and on ∂D, respectively, find a function $u(x)$ in D such that

$$\begin{cases} (A - \lambda)u = f & \text{in } D, \\ Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = \varphi & \text{on } \partial D. \end{cases} \tag{1.1}$$

Here:

(4) λ is a *complex* parameter.
(5) $\mu(x')$ and $\gamma(x')$ are real-valued, smooth functions on the boundary ∂D.
(6) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D (see Figure 1.1).

We remark that if $\mu(x') \neq 0$ and $\gamma(x') \equiv 0$ on ∂D (resp. $\mu(x') \equiv 0$ and $\gamma(x') \neq 0$ on ∂D), then the boundary condition L is essentially the so-called Neumann (resp. Dirichlet) condition.

It is easy to see that problem (1.1) is non-degenerate (or coercive) if and only if either $\mu(x') \neq 0$ on ∂D or $\mu(x') \equiv 0$ and $\gamma(x') \neq 0$ on ∂D. The generation theorem of analytic semigroups is well established in the non-degenerate case both in the L^p topology and in the topology of uniform convergence (cf. Friedman [Fr1], Tanabe [Tn], Masuda [Ma], Stewart [Sw]).

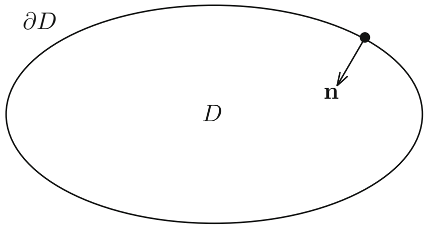

Fig. 1.1.

In this book, under the condition that $\mu(x') \geq 0$ on ∂D we shall consider the problem of existence and uniqueness of solutions of problem (1.1) in the framework of Sobolev spaces of L^p type, and generalize the generation theorem of analytic semigroups to the *degenerate* case.

First, we give a fundamental *a priori* estimate for problem (1.1).

If $1 \leq p < \infty$, we let

$$\begin{aligned} L^p(D) = \ & \text{the space of (equivalence classes of) Lebesgue} \\ & \text{measurable functions } u(x) \text{ on } D \text{ such that } |u(x)|^p \\ & \text{is integrable on } D. \end{aligned}$$

The space $L^p(D)$ is a Banach space with the norm

$$\|u\|_p = \left(\int_D |u(x)|^p dx \right)^{1/p}.$$

If m is a non-negative integer, we define the usual Sobolev space

$$\begin{aligned} W^{m,p}(D) = \ & \text{the space of (equivalence classes of) functions} \\ & u \in L^p(D) \text{ whose derivatives } D^\alpha u(x),\ |\alpha| \leq m, \\ & \text{in the sense of distributions are in } L^p(D). \end{aligned}$$

The space $W^{m,p}(D)$ is a Banach space with the norm

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leq m} \int_D |D^\alpha u(x)|^p dx \right)^{1/p}.$$

We remark that

$$W^{0,p}(D) = L^p(D); \quad \|\cdot\|_{0,p} = \|\cdot\|_p.$$

Furthermore, we let

$$\begin{aligned} B^{m-1/p,p}(\partial D) = \ & \text{the space of the boundary values } \varphi(x') \text{ of functions} \\ & u \in W^{m,p}(D). \end{aligned}$$

In the space $B^{m-1/p,p}(\partial D)$, we introduce a norm

$$|\varphi|_{m-1/p,p} = \inf \|u\|_{m,p},$$

where the infimum is taken over all functions $u \in W^{m,p}(D)$ which equal $\varphi(x')$ on the boundary ∂D. The space $B^{m-1/p,p}(\partial D)$ is a Banach space with respect to this norm $|\cdot|_{m-1/p,p}$; more precisely, it is a Besov space (cf. [BL], [Tb], [Tr]).

Then we have the following result:

Theorem 1.1. *Let $1 < p < \infty$. Assume that the following two conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ on ∂D.
(B) $\gamma(x') < 0$ on $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then, for any solution $u \in W^{2,p}(D)$ of problem (1.1) with $f \in L^p(D)$ and $\varphi \in B^{2-1/p,p}(\partial D)$ we have the a priori *estimate*

$$\|u\|_{2,p} \leq C(\lambda) \left(\|f\|_p + |\varphi|_{2-1/p,p} + \|u\|_p \right), \tag{1.2}$$

with a positive constant $C(\lambda)$ depending on λ.

It should be emphasized that problem (1.1) is a *degenerate* elliptic boundary value problem from an analytical point of view. This is due to the fact that the so-called Lopatinskii–Shapiro complementary condition is violated at each point x' of the set M. Amann [Am] studied the non-degenerate case; more precisely, he assumes that the boundary ∂D is the disjoint union of the two closed subsets $M = \{x' \in \partial D : \mu(x') = 0\}$ and $\partial D \setminus M = \{x' \in \partial D : \mu(x') > 0\}$, each of which is an $(N-1)$ dimensional compact smooth manifold.

Here it is worth while pointing out that the *a priori* estimate (1.2) is the same one for the Dirichlet condition: $\mu(x') \equiv 0$ and $\gamma(x') \neq 0$ on ∂D (cf. [ADN], [LM]).

Now we state a generation theorem of analytic semigroups in the L^p topology.

We associate with problem (1.1) a unbounded linear operator A_p from $L^p(D)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(A_p)$ of A_p is the set

$$\mathcal{D}(A_p) = \left\{ u \in W^{2,p}(D) : Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \right\}. \tag{1.3}$$

(b) $A_p u = Au$, $u \in \mathcal{D}(A_p)$.

The next theorem is an L^p version of Taira [Ta1, Theorem 1]:

Theorem 1.2. *Let $1 < p < \infty$. Assume that conditions (A) and (B) are satisfied. Then we have the following two assertions:*

(i) *For every positive number* ε, *there exists a positive constant* $r_p(\varepsilon)$ *such that the resolvent set of* A_p *contains the set*

$$\Sigma_p(\varepsilon) = \left\{\lambda = r^2 e^{i\theta} : r \geq r_p(\varepsilon),\ -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$

and that the resolvent $(A_p - \lambda I)^{-1}$ *satisfies the estimate*

$$\left\|(A_p - \lambda I)^{-1}\right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon), \tag{1.4}$$

where $c_p(\varepsilon)$ *is a positive constant depending on* ε.

(ii) *The operator* A_p *generates a semigroup* U_z *on the space* $L^p(D)$ *which is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$ *(see Figure 1.2).*

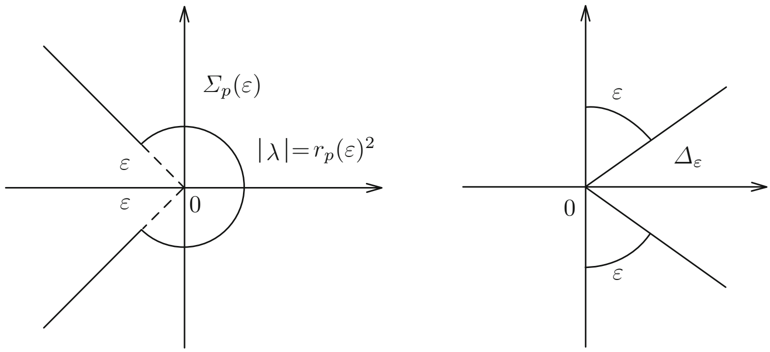

Fig. 1.2.

Secondly, we state a generation theorem of analytic semigroups in the topology of uniform convergence.

Let $C(\overline{D})$ be the space of real-valued, continuous functions $f(x)$ on $\overline{D}$. We equip the space $C(\overline{D})$ with the topology of uniform convergence on the whole $\overline{D}$; hence it is a Banach space with the maximum norm

$$\|f\|_\infty = \max_{x \in \overline{D}} |f(x)|.$$

We introduce a subspace of $C(\overline{D})$ which is associated with the boundary condition L. We remark that the boundary condition

$$Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 \quad \text{on } \partial D$$

includes the condition

$$u = 0 \quad \text{on } M = \{x' \in \partial D : \mu(x') = 0\},$$

if $\gamma(x') \neq 0$ on M. With this fact in mind, we let

$$C_0(\overline{D} \setminus M) = \left\{u \in C(\overline{D}) : u = 0 \text{ on } M\right\}.$$

The space $C_0(\overline{D} \setminus M)$ is a closed subspace of $C(\overline{D})$; hence it is a Banach space. Furthermore, we introduce a unbounded linear operator $\mathfrak{A}$ from $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the set

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ Lu = 0\right\}. \tag{1.5}$$

(b) $\mathfrak{A}u = Au$, $u \in \mathcal{D}(\mathfrak{A})$.
Here Au and Lu are taken in the sense of *distributions* (see Chapter 9).

Then Theorem 1.2 remains valid with $L^p(D)$ and A_p replaced by $C_0(\overline{D} \setminus M)$ and $\mathfrak{A}$, respectively. More precisely, we can prove the following:

Theorem 1.3. *Assume that condition (A) and the following condition (B′) (replacing condition (B)) are satisfied:*

(B′) $\gamma(x') \leq 0$ *on* ∂D *and* $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then we have the following two assertions:

(i) For every positive number ε*, there exists a positive constant* $r(\varepsilon)$ *such that the resolvent set of* $\mathfrak{A}$ *contains the set*

$$\Sigma(\varepsilon) = \left\{\lambda = r^2 e^{i\theta} : r \geq r(\varepsilon),\ -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$

and that the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ *satisfies the estimate*

$$\|(\mathfrak{A} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon), \tag{1.6}$$

where $c(\varepsilon)$ *is a positive constant depending on* ε.

(ii) The operator $\mathfrak{A}$ *generates a semigroup* T_z *on the space* $C_0(\overline{D} \setminus M)$ *which is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$ *(see Figure 1.3).*

Moreover, the operators T_t $(t \geq 0)$ *are non-negative and contractive on the space* $C_0(\overline{D} \setminus M)$*:*

$$f \in C_0(\overline{D} \setminus M),\ 0 \leq f(x) \leq 1 \text{ on } \overline{D} \setminus M \quad \Longrightarrow \quad 0 \leq T_t f(x) \leq 1 \text{ on } \overline{D} \setminus M.$$

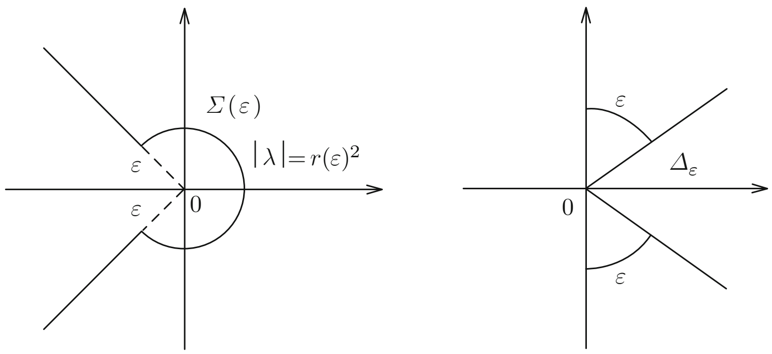

Fig. 1.3.

The main purpose of this book is devoted to the functional analytic approach to the problem of existence of Markov processes in probability theory.

A strongly continuous, non-negative and contraction semigroup $\{T_t\}_{t\geq 0}$ on the space $C_0(\overline{D} \setminus M)$ is called a *Feller semigroup* on $\overline{D} \setminus M$.

Therefore, we can reformulate part (ii) of Theorem 1.3 as follows:

Theorem 1.4. *Assume that conditions (A) and (B′) are satisfied. Then the operator* $\mathfrak{A}$ *generates a Feller semigroup* $\{T_t\}_{t\geq 0}$ *on the space* $\overline{D} \setminus M$.

Theorem 1.4 generalizes Bony–Courrège–Priouret [BCP, Théorème XIX] to the case where $\mu(x') \geq 0$ on the boundary ∂D (cf. [Ta2, Theorem 10.1.3]).

It is known (cf. [Dy2], [Ta2, Chapter 9]) that if $\{T_t\}_{t\geq 0}$ is a Feller semigroup on $\overline{D} \setminus M$, then there exists a unique Markov transition function $p_t(x, \cdot)$ on the space $\overline{D} \setminus M$ such that

$$T_t f(x) = \int_{\overline{D}\setminus M} p_t(x, dy) f(y), \quad f \in C_0(\overline{D} \setminus M).$$

Furthermore, it can be shown that the function $p_t(x, \cdot)$ is the transition function of some strong *Markov process* $\mathcal{X}$; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

The differential operator A describes analytically a strong Markov process with continuous paths in the interior D such as Brownian motion (see Figure 1.4).

The terms $\mu(x')\partial u/\partial \mathbf{n}$ and $\gamma(x')u$ of the boundary condition L are supposed to correspond to reflection and absorption phenomena, respectively. The situation may be represented schematically by Figure 1.5.

Hence the intuitive meaning of condition (B′) is that absorption phenomenon occurs at each point of the set $M = \{x' \in \partial D : \mu(x') = 0\}$, while reflection phenomenon occurs at each point of the set $\partial D \setminus M = \{x' \in \partial D : \mu(x') > 0\}$. In other words, a Markovian particle moves in the space $\overline{D} \setminus M$

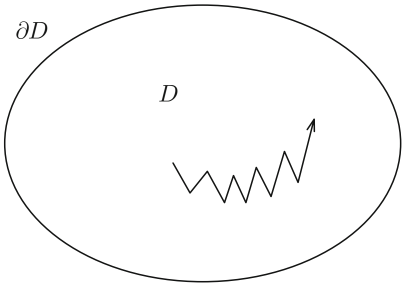

Fig. 1.4.

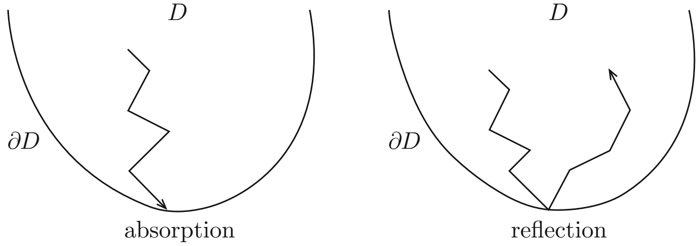

Fig. 1.5.

until it "dies" at the time when it reaches the set M where the particle is definitely absorbed (see Figure1.6). Therefore, Theorem 1.4 asserts that there exists a Feller semigroup corresponding to such a diffusion phenomenon.

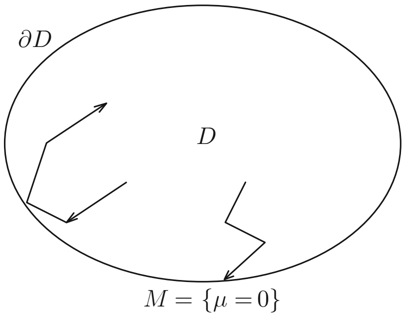

Fig. 1.6.

It is worth while pointing out here that the condition

$$\mu(x') \geq 0 \text{ and } \gamma(x') \leq 0 \quad \text{on } \partial D$$

is necessary in order that the operator $\mathfrak{A}$ is the infinitesimal generator of a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D} \setminus M$ (cf. [Ta2, Section 9.5]).

As an application of Theorem 1.2, we consider the following *semilinear* initial-boundary value problem: Given functions $f(x,t,u,\xi)$ and $u_0(x)$ defined in $D\times[0,T)\times\mathbf{R}\times\mathbf{R}^N$ and in D, respectively, find a function $u(x,t)$ in $D\times[0,T)$ such that

$$\begin{cases} \left(\frac{\partial}{\partial t} - A\right) u(x,t) = f(x,t,u,\operatorname{grad} u) & \text{in } D \times (0,T), \\ Lu(x',t) = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 & \text{on } \partial D \times [0,T), \\ u(x,0) = u_0(x) & \text{in } D. \end{cases} \tag{1.7}$$

By making use of the operator A_p, we can formulate problem (1.7) in terms of the abstract *Cauchy problem* in the Banach space $L^p(D)$ as follows:

$$\begin{cases} \frac{du}{dt} = A_p u(t) + F\left(t, u(t)\right), \quad 0 < t < T, \\ u|_{t=0} = u_0. \end{cases} \tag{1.8}$$

Here $u(t) = u(\cdot,t)$ and $F\left(t,u(t)\right) = f\left(\cdot,t,u(t),\operatorname{grad} u(t)\right)$ are functions defined on the interval $[0,T)$, taking values in the space $L^p(D)$.

We can prove local existence and uniqueness theorems for problem (1.8) (Theorems 10.1 and 10.2), by using the theory of fractional powers of analytic semigroups. Our semigroup approach here can be traced back to the pioneering work of Fujita–Kato [FK] on the Navier–Stokes equation in fluid dynamics.

Theorem 1.5. *Assume that conditions (A) and (B) are satisfied. If the non-linear term $f(x,t,u,\xi)$ is a locally Lipschitz continuous function with respect to all its variables $(x,t,u,\xi) \in D\times[0,T)\times\mathbf{R}\times\mathbf{R}^N$ with the possible exception of the x variables, then we have the following two assertions:*

(i) If $N < p < \infty$, then, for every function $u_0(x)$ of $\mathcal{D}(A_p)$, problem (1.8) has a unique local solution $u(x,t) \in C\left([0,T_1];\, L^p(D)\right) \cap C^1\left((0,T_1);\, L^p(D)\right)$ where $T_1 = T_1(p,u_0)$ is a positive constant.

(ii) If $N/2 < p < N$, we assume that there exist a non-negative continuous function $\rho(t,r)$ on $\mathbf{R}\times\mathbf{R}$ and a constant $1 \le \gamma < N/(N-p)$ such that the following four conditions are satisfied:
(a) $|f(x,t,u,\xi)| \le \rho(t,|u|)(1+|\xi|^\gamma)$.
(b) $|f(x,t,u,\xi) - f(x,s,u,\xi)| \le \rho(t,|u|)\left(1+|\xi|^\gamma\right)|t-s|$.
(c) $|f(x,t,u,\xi) - f(x,t,u,\eta)| \le \rho(t,|u|)\left(1+|\xi|^{\gamma-1}+|\eta|^{\gamma-1}\right)|\xi-\eta|$.
(d) $|f(x,t,u,\xi) - f(x,t,v,\xi)| \le \rho(t,|u|+|v|)\left(1+|\xi|^\gamma\right)|u-v|$.
Then, for every function $u_0(x)$ of $\mathcal{D}(A_p)$, problem (1.8) has a unique local solution $u(x,t) \in C\left([0,T_2];\, L^p(D)\right) \cap C^1\left((0,T_2);\, L^p(D)\right)$ where $T_2 = T_2(p,u_0)$ is a positive constant.

Here $C\left([0,T];\, L^p(D)\right)$ denotes the space of continuous functions on the closed interval $[0,T]$ taking values in $L^p(D)$, and $C^1\left((0,T);\, L^p(D)\right)$ denotes the space of continuously differentiable functions on the open interval $(0,T)$ taking values in $L^p(D)$, respectively.

The rest of this monograph is organized as follows.

Chapter 2 is devoted to a review of standard topics from the theory of semigroups. Section 2.1 provides a brief description of the basic results about analytic semigroups (Theorem 2.2) which forms a functional analytic background for the proof of Theorems 1.2 and 1.3. Moreover, Subsection 2.1.3 is devoted to the semigroup approach to a class of initial-boundary value problems for semilinear parabolic differential equations. By making use of fractional powers of analytic semigroups, we formulate a local existence and uniqueness theorem for semilinear initial-boundary value problems (Theorem 2.8).

On the other hand, Section 2.2 provides a brief description of basic definitions and results about Markov processes and Feller semigroups. In Subsection 2.2.6 we prove various generation theorems of Feller semigroups by using the Hille–Yosida theory of semigroups (Theorems 2.16 and 2.18) which form a functional analytic background for the proof of Theorem 1.4.

In Chapter 3 we present a brief description of the basic concepts and results of the L^p theory of pseudo-differential operators which may be considered as a modern theory of the classical potential theory. In particular, we formulate the Besov space boundedness theorem due to Bourdaud [Bo] (Theorem 3.15) and a useful criterion for hypoellipticity due to Hörmander [Ho2] (Theorem 3.16) which play an essential role in the proof of our main results.

In Chapter 4 we study the boundary value problem (1.1) in the framework of Sobolev spaces of L^p type, by using the L^p theory of pseudo-differential operators. The idea of our approach is stated as follows:

First, we consider the following *Neumann* problem:

$$\begin{cases} (A-\lambda)v = f & \text{in } D, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases} \tag{1.9}$$

The existence and uniqueness theorem for problem (1.9) is well established in the framework of Sobolev spaces of L^p type (cf. Agmon–Douglis–Nirenberg [ADN]). We let

$$v = G_N(\lambda) f.$$

The operator $G_N(\lambda)$ is the Green operator for the Neumann problem. Then it follows that a function $u(x)$ is a solution of problem (1.1) if and only if the function $w(x) = u(x) - v(x)$ is a solution of the problem

$$\begin{cases} (A-\lambda)w = 0 & \text{in } D, \\ Lw = -Lv = -\mu(x')\frac{\partial v}{\partial \mathbf{n}} - \gamma(x')v = -\gamma(x')v & \text{on } \partial D. \end{cases}$$

However, we know that every solution w of the homogeneous equation

$$(A-\lambda)w = 0 \quad \text{in } D$$

can be expressed by means of a single layer potential as follows:

$$w = P(\lambda)\psi.$$

The operator $P(\lambda)$ is the Poisson operator for the Dirichlet problem. Thus, by using the operators $G_N(\lambda)$ and $P(\lambda)$ we can reduce the study of problem (1.1) to that of the equation

$$T(\lambda)\psi = LP(\lambda)\psi = -\gamma(x')v, \quad v = G_N(\lambda)f.$$

This is a generalization of the classical Fredholm integral equation.

It is well known (cf. [Ho1], [Ho3], [Se2], [Ty]) that the operator $T(\lambda) = LP(\lambda)$ is a pseudo-differential operator of first order on the boundary ∂D. We can prove that the *a priori* estimate (1.2) of Theorem 1.1 is entirely equivalent to the corresponding *a priori* estimate for the operator $T(\lambda)$ (Theorem 4.10).

Chapter 5 is devoted to the proof of Theorem 1.1. We study the pseudo-differential operator $T(\lambda)$ in question, and prove that conditions (A) and (B) are sufficient for the validity of the *a priori* estimate (1.2) (Lemma 5.1). More precisely, we construct a *parametrix* $S(\lambda)$ for $T(\lambda)$ in the Hörmander class $L^0_{1,1/2}(\partial D)$ (Lemma 5.2), and then apply a *Besov space boundedness theorem* (Theorem 3.15) to the parametrix $S(\lambda)$ to obtain the *a priori* estimate (1.2) for problem (1.1).

Here it should be emphasized that if we use instead of the Neumann problem (1.9) the *Dirichlet* problem as usual, then we have the following *a priori* estimate for problem (1.1):

$$\|u\|_{1,p} \leq C(\lambda) \left(\|f\|_p + |\varphi|_{1-1/p,p} + \|u\|_p \right).$$

In other words, we can *not* use this estimate to prove the generation theorem of analytic semigroups in the L^p topology.

In Chapter 6 we study the operator A_p, and prove fundamental *a priori* estimates for $A_p - \lambda I$ (Theorem 6.3) which is an essential step in the proof of Theorem 1.2. We make good use of Agmon's method (Proposition 6.4). This is a technique of treating a spectral parameter λ as a second-order, elliptic differential operator of an extra variable and relating the old problem to a new one with the additional variable.

Chapter 7 is devoted to the proof of Theorem 1.2 (Theorems 7.1 and 7.9). Once again we make use of Agmon's method in the proof of Theorems 7.1 and 7.9. In particular, Agmon's method plays a fundamental role in the proof of the *surjectivity* of the operator $A_p - \lambda I$ (Proposition 7.2).

Chapter 8 and Chapter 9 are devoted to the proof of Theorem 1.3 and Theorem 1.4. In Chapter 8 we prove part (i) of Theorem 1.3. Part (i) of Theorem 1.3 follows from Theorem 1.2 by using Sobolev's imbedding theorems (Theorems 8.1 and 8.2) and a λ-*dependent localization* argument essentially due to Masuda [Ma] (Lemma 8.4).

In Chapter 9 we prove Theorem 1.4 and part (ii) of Theorem 1.3. This chapter is the heart of the subject. General existence theorems for Feller semigroups are formulated in terms of elliptic boundary value problems with spectral parameter (Theorem 9.12). First, we study Feller semigroups with reflecting barrier (Theorem 9.14) and then, by using these Feller semigroups we construct Feller semigroups corresponding to such a diffusion phenomenon that either absorption or reflection phenomenon occurs at each point of the boundary (Theorem 9.18). Our proof is based on the generation theorems of Feller semigroups discussed in Section 2.2.

In Chapter 10 we study problem (1.8), and prove Theorem 1.5 by using the theory of fractional powers of analytic semigroups (Theorems 10.1 and 10.2). To do this, we verify that all the conditions of Theorem 2.8 are satisfied. We remark that Theorem 1.5 is a generalization of Pazy [Pa, Section 8.4, Theorems 4.4 and 4.5] to the degenerate case.

In the final Chapter 11, as concluding remarks, we give an overview for general results on generation theorems for Feller semigroups proved mainly by the author using the theory of pseudo-differential operators ([Ho1], [Se1], [Se2]) and the Calderón–Zygmund theory of singular integral operators ([CZ]).

In Appendix A, we formulate various maximum principles for second-order elliptic differential operators such as the weak maximum principle (Theorem A.1) and the Hopf boundary point lemma (Lemma A.3) which play an important role in Chapter 9.

The following diagram gives a bird's eye view of Markov processes, Feller semigroups and boundary value problems and how these relate to each other:

Probability	**Functional Analysis**	**Partial Differential Equations**
Markov process $\mathcal{X}$	Feller semigroup $\{T_t\}$	infinitesimal generator $\mathfrak{A}$
Markov transition function $p_t(\cdot, dy)$	$T_t f = \int_{\overline{D}} p_t(\cdot, dy) f(y)$	$T_t = \exp[t\mathfrak{A}]$
Chapman and Kolmogorov equation	semigroup property $T_{t+s} = T_t \cdot T_s$	differential operator A
absorption and reflection phenomena	function space $C_0(\overline{D} \setminus M)$	boundary condition L

2

Semigroup Theory

This chapter is devoted to a review of standard topics from the theory of semigroups which forms a functional analytic background for the proof of Theorems 1.2, 1.3, 1.4 and 1.5.

2.1 Analytic Semigroups

This section provides a brief description of the basic results of the theory of analytic semigroups which forms a functional analytic background for the proof of Theorems 1.2 and 1.3. Moreover, Subsection 2.1.3 is devoted to the semigroup approach to a class of initial-boundary value problems for semilinear parabolic differential equations (Theorem 2.8). Theorem 1.5 follows by verifying all the conditions of Theorem 2.8. For more leisurely treatments of analytic semigroups, the reader is referred to Friedman [Fr1], Pazy [Pa], Tanabe [Tn], Yosida [Yo] and also Taira [Ta4].

2.1.1 Generation of Analytic Semigroups

Let E be a Banach space over the real or complex number field, and let $A : E \to E$ be a *densely defined*, closed linear operator with domain $\mathcal{D}(A)$.

Assume that the operator A satisfies the following two conditions (see Figure 2.1 below):

(1) The resolvent set of A contains the region

$$\Sigma_\omega = \{\lambda \in \mathbf{C} : \lambda \neq 0, |\arg \lambda| < \pi/2 + \omega\}, \quad 0 < \omega < \pi/2.$$

(2) For each $\varepsilon > 0$, there exists a positive constant M_ε such that the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ satisfies the estimate

$$\|R(\lambda)\| \leq \frac{M_\varepsilon}{|\lambda|} \text{ for all } \lambda \in \Sigma_\omega^\varepsilon = \{\lambda \in \mathbf{C} : \lambda \neq 0, \ |\arg \lambda| \leq \pi/2 + \omega - \varepsilon\}. \tag{2.1}$$

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Lecture Notes in Mathematics 1499, DOI 10.1007/978-3-642-01677-6_2,

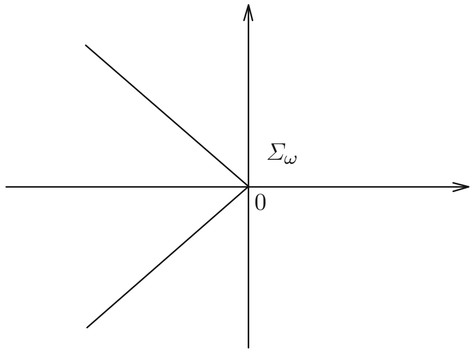

Fig. 2.1.

Then we let

$$U(t) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda)\,d\lambda. \tag{2.2}$$

Here Γ is a path in the set $\Sigma_\omega^\varepsilon$ consisting of the following three curves (see Figure 2.2):

$$\Gamma^{(1)} = \left\{re^{-i(\pi/2+\omega-\varepsilon)} : 1 \le r < \infty\right\},$$
$$\Gamma^{(2)} = \left\{e^{i\theta} : -(\pi/2+\omega-\varepsilon) \le \theta \le \pi/2+\omega-\varepsilon\right\},$$
$$\Gamma^{(3)} = \left\{re^{i(\pi/2+\omega-\varepsilon)} : 1 \le r < \infty\right\}.$$

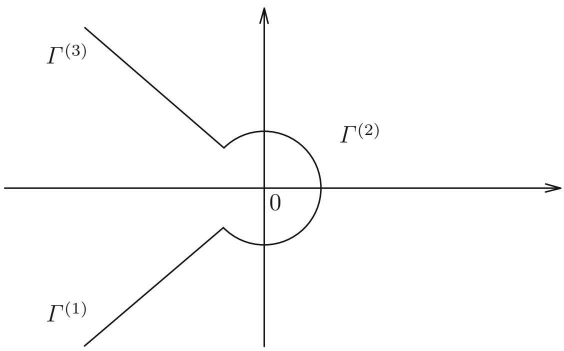

Fig. 2.2.

It is easy to see that the integral

$$U(t) = -\frac{1}{2\pi i}\sum_{k=1}^{3}\int_{\Gamma^{(k)}} e^{\lambda t}R(\lambda)\,d\lambda$$

converges in the uniform operator topology of the Banach space $L(E,E)$ for all $t > 0$, and thus defines a bounded linear operator on E. Here $L(E,E)$ denotes the space of bounded linear operators on E.

Furthermore, we have the following:

Proposition 2.1. *The operators $U(t)$, defined by formula (2.2), form a* semigroup *on E, that is, they enjoy the* semigroup property

$$U(t+s) = U(t) \cdot U(s) \quad \text{for all } t,\ s > 0.$$

Proof. By Cauchy's theorem, we may assume that

$$U(s) = -\frac{1}{2\pi i} \int_{\Gamma'} e^{\mu s} R(\mu)\, d\mu, \quad s > 0.$$

Here Γ' is a path obtained from the path Γ by translating each point of Γ to the right by a fixed small positive distance (see Figure 2.3).

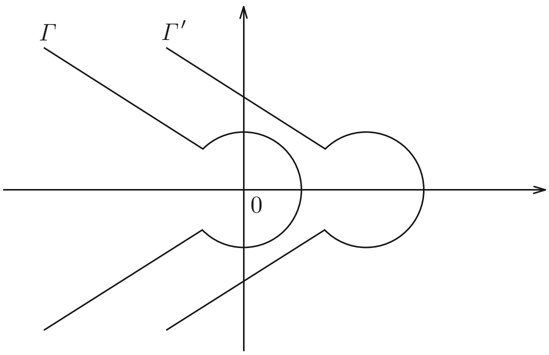

Fig. 2.3.

Then we have, by Fubini's theorem,

$$\begin{aligned} U(t) \cdot U(s) &= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} e^{\lambda t} e^{\mu s} R(\lambda)\, R(\mu)\, d\lambda\, d\mu \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} e^{\lambda t} e^{\mu s} \frac{R(\lambda) - R(\mu)}{\lambda - \mu}\, d\lambda\, d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} R(\lambda) \left[\frac{1}{2\pi i} \int_{\Gamma'} \frac{e^{\mu s}}{\lambda - \mu}\, d\mu \right] d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma'} e^{\mu s} R(\mu) \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{e^{\lambda t}}{\lambda - \mu}\, d\lambda \right] d\mu. \end{aligned}$$

We calculate the two terms in the last part.

(a) We let

$$f(\mu) = \frac{e^{\mu s}}{\lambda - \mu}, \quad \mu \in \mathbf{C}.$$

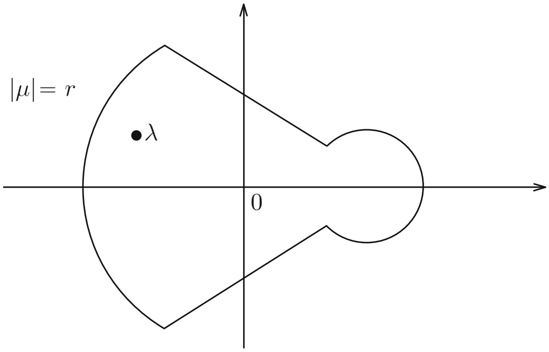

Fig. 2.4.

Then, by applying the residue theorem we obtain that (see Figure 2.4)

$$\begin{aligned}
&\int_{\Gamma'^{(1)}\cap\{|\mu|\le r\}} f(\mu)\,d\mu + \int_{\Gamma'^{(2)}} f(\mu)\,d\mu + \int_{\Gamma'^{(3)}\cap\{|\mu|\le r\}} f(\mu)\,d\mu \\
&\quad + \int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon} f(re^{i\theta}) rie^{i\theta}\,d\theta \\
&= 2\pi i \operatorname{Res}\,[f(\mu)]_{\mu=\lambda} \\
&= -2\pi i e^{\lambda s}.
\end{aligned}$$

However, we have, as $r \to \infty$,

$$\begin{aligned}
\int_{\Gamma'^{(1)}\cap\{|\mu|\le r\}} f(\mu)\,d\mu &\longrightarrow \int_{\Gamma'^{(1)}} f(\mu)\,d\mu, \\
\int_{\Gamma'^{(3)}\cap\{|\mu|\le r\}} f(\mu)\,d\mu &\longrightarrow \int_{\Gamma'^{(3)}} f(\mu)\,d\mu,
\end{aligned}$$

and

$$\left| \int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon} f(re^{i\theta}) rie^{i\theta}\,d\theta \right| \le e^{-rs\cdot\sin(\omega-\varepsilon)} \int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon} \frac{d\theta}{\left|\frac{\lambda}{r} - e^{i\theta}\right|} \longrightarrow 0.$$

Therefore, we find that

$$\frac{1}{2\pi i}\int_{\Gamma'} \frac{e^{\mu s}}{\lambda - \mu}\,d\mu = -e^{\lambda s}.$$

(b) Similarly, since the path Γ lies to the left of the path Γ', we find that

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{e^{\lambda t}}{\lambda - \mu}\,d\lambda = 0.$$

Summing up, we obtain that

$$U(t) \cdot U(s) = -\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda(t+s)} R(\lambda)\, d\lambda = U(t+s) \quad \text{for all } t,\ s > 0.$$

The proof of Proposition 2.1 is complete. □

The next theorem states that the semigroup $U(t)$ can be extended to an analytic semigroup in some sector containing the positive real axis.

Theorem 2.2. *The semigroup $U(t)$, defined by formula (2.2), can be extended to a semigroup $U(z)$ which is* analytic *in the sector*

$$\Delta_\omega = \{z = t + is : z \neq 0, |\arg z| < \omega\},$$

and enjoys the following properties:

(a) The operators $AU(z)$ and $\frac{dU}{dz}(z)$ are bounded operators on E for each $z \in \Delta_\omega$, and satisfy the relation

$$\frac{dU}{dz}(z) = AU(z) \quad \textit{for all } z \in \Delta_\omega. \tag{2.3}$$

(b) For each $0 < \varepsilon < \omega/2$, there exist positive constants $\widetilde{M_0}(\varepsilon)$ and $\widetilde{M_1}(\varepsilon)$ such that

$$\|U(z)\| \leq \widetilde{M_0}(\varepsilon) \qquad \textit{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.4}$$

$$\|AU(z)\| \leq \frac{\widetilde{M_1}(\varepsilon)}{|z|} \quad \textit{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.5}$$

where (see Figure 2.5)

$$\Delta_\omega^{2\varepsilon} = \{z \in \mathbf{C} : z \neq 0, |\arg z| \leq \omega - 2\varepsilon\}.$$

(c) For each $x \in E$, we have, as $z \to 0$, $z \in \Delta_\omega^{2\varepsilon}$,

$$U(z)x \longrightarrow x \quad \textit{in } E.$$

Proof. (i) The *analyticity* of $U(z)$: If $\lambda \in \Gamma^{(3)}$ and $z \in \Delta_\omega^{2\varepsilon}$, that is, if we have the formulas

$$\lambda = |\lambda| e^{i\theta}, \quad \theta = \pi/2 + \omega - \varepsilon,$$
$$z = |z| e^{i\varphi}, \quad |\varphi| \leq \omega - 2\varepsilon,$$

then it follows that

$$\lambda z = |\lambda|\,|z| e^{i(\theta+\varphi)},$$

with

$$\pi/2 + \varepsilon \leq \theta + \varphi \leq \pi/2 + 2\omega - 3\varepsilon < 3\pi/2 - 3\varepsilon.$$

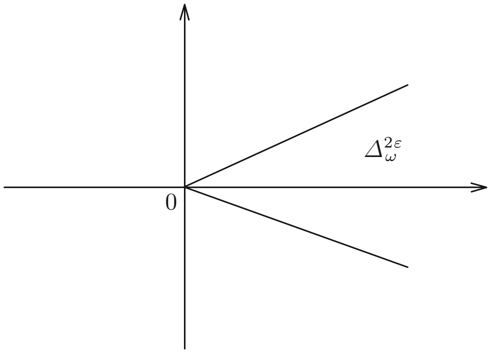

Fig. 2.5.

Note that

$$\cos(\theta + \varphi) \le \cos(\pi/2 + \varepsilon) = -\sin\varepsilon.$$

Hence we have the inequality

$$|e^{\lambda z}| \le e^{-|\lambda|\,|z|\sin\varepsilon} \quad \text{for all } \lambda \in \Gamma^{(3)} \text{ and } z \in \Delta_\omega^{2\varepsilon}. \tag{2.6}$$

Similarly, we have the inequality

$$|e^{\lambda z}| \le e^{-|\lambda|\,|z|\sin\varepsilon} \quad \text{for all } \lambda \in \Gamma^{(1)} \text{ and } z \in \Delta_\omega^{2\varepsilon}. \tag{2.7}$$

For each small $\varepsilon > 0$, we let

$$K_\omega^\varepsilon = \Delta_\omega^{2\varepsilon} \bigcap \{z \in \mathbf{C} : |z| \ge \varepsilon\} = \{z \in \mathbf{C} : |z| \ge \varepsilon, |\arg z| \le \omega - 2\varepsilon\}.$$

Then, by combining estimates (2.1), (2.6) and (2.7) we obtain that

$$\left\|e^{\lambda z} R(\lambda)\right\| \le \frac{M_\varepsilon}{|\lambda|} e^{-\varepsilon\sin\varepsilon\cdot|\lambda|} \quad \text{for all } \lambda \in \Gamma^{(1)} \bigcup \Gamma^{(3)} \text{ and } z \in K_\omega^\varepsilon. \tag{2.8}$$

On the other hand, we have the estimate

$$\left\|e^{\lambda z} R(\lambda)\right\| \le M_\varepsilon e^{|z|} \quad \text{for all } \lambda \in \Gamma^{(2)} \text{ and } z \in K_\omega^\varepsilon. \tag{2.9}$$

Therefore, we find that the integral

$$U(z) = -\frac{1}{2\pi i} \int_\Gamma e^{\lambda z} R(\lambda)\, d\lambda = -\frac{1}{2\pi i} \sum_{k=1}^{3} \int_{\Gamma^{(k)}} e^{\lambda z} R(\lambda)\, d\lambda \tag{2.10}$$

converges in the Banach space $L(E, E)$, uniformly in $z \in K_\omega^\varepsilon$, for every $\varepsilon > 0$. This proves that the operator $U(z)$ is analytic in the domain $\Delta_\omega = \bigcup_{\varepsilon>0} K_\omega^\varepsilon$.

By the analyticity of $U(z)$, it follows that the operators $U(z)$ also enjoy the semigroup property

$$U(z+w) = U(z) \cdot U(w) \quad \text{for all } z,\, w \in \Delta_\omega.$$

(ii) We prove that the operators $U(z)$ enjoy properties (a), (b) and (c).
(b) First, by using Cauchy's theorem we obtain that

$$U(z) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z} R(\lambda)\, d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_{|z|}} e^{\lambda z} R(\lambda)\, d\lambda,$$

where $\Gamma_{|z|}$ is a path consisting of the following three curves (see Figure 2.6):

$$\Gamma_{|z|}^{(1)} = \left\{ r e^{-i(\pi/2+\omega-\varepsilon)} : \frac{1}{|z|} \leq r < \infty \right\},$$

$$\Gamma_{|z|}^{(2)} = \left\{ \frac{1}{|z|} e^{i\theta} : -(\pi/2+\omega-\varepsilon) \leq \theta \leq \pi/2+\omega-\varepsilon \right\},$$

$$\Gamma_{|z|}^{(3)} = \left\{ r e^{i(\pi/2+\omega-\varepsilon)} : \frac{1}{|z|} \leq r < \infty \right\}.$$

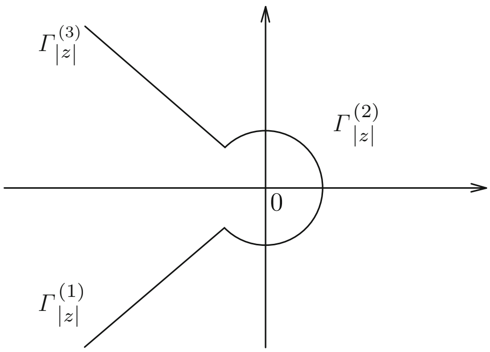

Fig. 2.6.

However, by estimates (2.1), (2.6) and (2.7), it follows that

$$\left\| e^{\lambda z} R(\lambda) \right\| \leq \frac{M_\varepsilon}{|\lambda|} e^{-|\lambda|\,|z| \sin\varepsilon} \quad \text{for all } \lambda \in \Gamma_{|z|}^{(1)} \bigcup \Gamma_{|z|}^{(3)} \text{ and } z \in \Delta_\omega^{2\varepsilon}.$$

Hence we have, for $k = 1, 3$,

$$\begin{aligned}\int_{\Gamma_{|z|}^{(k)}} \left\| e^{\lambda z} R(\lambda) \right\| |\, d\lambda| &\leq M_\varepsilon \int_{\frac{1}{|z|}}^\infty e^{-\rho |z| \sin\varepsilon} \rho^{-1}\, d\rho \\ &= M_\varepsilon \int_1^\infty e^{-\sin\varepsilon \cdot s} s^{-1}\, ds.\end{aligned}$$

We have also, for $k = 2$,

$$\int_{\Gamma_{|z|}^{(2)}} \left\| e^{\lambda z} R(\lambda) \right\| |\, d\lambda| \leq M_\varepsilon \int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon} e \, d\theta$$
$$= 2eM_\varepsilon(\pi/2 + \omega - \varepsilon)$$
$$\leq 2\pi e M_\varepsilon.$$

Summing up, we obtain the following estimate:

$$\|U(z)\| \leq \frac{1}{2\pi} \sum_{k=1}^{3} \int_{\Gamma_{|z|}^{(k)}} \left\| e^{\lambda z} R(\lambda) \right\| |\, d\lambda|$$
$$\leq \frac{1}{2\pi} \left(2M_\varepsilon \int_1^\infty s^{-1} e^{-\sin\varepsilon \cdot s} \, ds + 2\pi e M_\varepsilon \right)$$
$$= \frac{M_\varepsilon}{\pi} \left(\int_1^\infty s^{-1} e^{-\sin\varepsilon \cdot s} \, ds + \pi e \right).$$

This proves the desired estimate (2.4), with

$$\widetilde{M_0}(\varepsilon) = \frac{M_\varepsilon}{\pi} \left(\int_1^\infty s^{-1} e^{-\sin\varepsilon \cdot s} \, ds + \pi e \right).$$

To prove estimate (2.5), note that

$$AR(\lambda) = (A - \lambda I + \lambda I)R(\lambda) = I + \lambda R(\lambda),$$

so that

$$\|AR(\lambda)\| \leq 1 + M_\varepsilon \quad \text{for all } \lambda \in \Sigma_\omega^\varepsilon.$$

Hence, by arguing just as in the proof of estimate (2.4) we obtain that

$$\left\| \int_\Gamma e^{\lambda z} AR(\lambda) \, d\lambda \right\| \leq 2 \int_{\frac{1}{|z|}}^\infty e^{-\rho|z|\sin\varepsilon} (1 + M_\varepsilon) \, d\rho$$
$$+ \int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon} (1 + M_\varepsilon) \, e \frac{1}{|z|} \, d\theta$$
$$\leq 2 (1 + M_\varepsilon) \left(\int_1^\infty e^{-\sin\varepsilon \cdot s} \, ds + \pi e \right) \frac{1}{|z|}$$
$$\text{for all } z \in \Delta_\omega^{2\varepsilon}. \tag{2.11}$$

This proves that the integral

$$\int_\Gamma e^{\lambda z} AR(\lambda) \, d\lambda$$

is convergent in the Banach space $L(E, E)$, for every $z \in \Delta_\omega^{2\varepsilon}$. By the closedness of A, it follows that

$$U(z) \in \mathcal{D}(A) \quad \text{for all } z \in \Delta_\omega^{2\varepsilon},$$

and

$$AU(z) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z} AR(\lambda)\,d\lambda \quad \text{for all } z \in \Delta_\omega^{2\varepsilon}. \tag{2.12}$$

Therefore, the desired estimate (2.5) follows from estimate (2.11), with

$$\widetilde{M_1}(\varepsilon) = \frac{1+M_\varepsilon}{\pi}\left(\int_1^\infty e^{-\sin\varepsilon\cdot s}\,ds + \pi e\right).$$

We remark that formula (2.12) remains valid for all $z \in \Delta_\omega$, since $\Delta_\omega = \bigcup_{\varepsilon>0}\Delta_\omega^{2\varepsilon}$.

(a) By estimates (2.8) and (2.9), we can differentiate formula (2.10) under the integral sign to obtain that

$$\frac{dU}{dz}(z) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda R(\lambda)\,d\lambda \quad \text{for all } z \in \Delta_\omega. \tag{2.13}$$

On the other hand, it follows from formula (2.12) that

$$\begin{aligned} AU(z) &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z} AR(\lambda)\,d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}(I+\lambda R(\lambda))\,d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda R(\lambda)\,d\lambda \quad \text{for all } z \in \Delta_\omega, \end{aligned} \tag{2.14}$$

since we have, by Cauchy's theorem,

$$\int_\Gamma e^{\lambda z}\,d\lambda = 0.$$

Therefore, the desired formula (2.3) follows immediately from formulas (2.13) and (2.14).

(c) Now let x_0 be an arbitrary element of $\mathcal{D}(A)$. By the residue theorem, it follows that

$$x_0 = \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda z}}{\lambda} x_0\,d\lambda.$$

Hence we have the formula

$$\begin{aligned} U(z)x_0 - x_0 &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\left(R(\lambda)+\frac{1}{\lambda}\right)x_0\,d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda z}}{\lambda} R(\lambda)Ax_0\,d\lambda. \end{aligned}$$

Here we remark that

$$\begin{aligned} &\left\|\frac{1}{\lambda}R(\lambda)\right\| \le \frac{M_\varepsilon}{|\lambda|^2} \quad \text{for all } \lambda \in \Gamma, \\ &\left|e^{\lambda z}\right| \le 2e^{-|\lambda|\,|z|\sin\varepsilon} + e^{|z|} \quad \text{for all } z \in \Delta_\omega^{2\varepsilon} \text{ and } \lambda \in \Gamma. \end{aligned}$$

Thus it follows from an application of the Lebesgue dominated convergence theorem that, as $z \to 0$, $z \in \Delta_\omega^{2\varepsilon}$,

$$U(z)x_0 - x_0 \longrightarrow -\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda} R(\lambda) A x_0 \, d\lambda.$$

However, we have the assertion

$$\int_\Gamma \frac{1}{\lambda} R(\lambda) A x_0 \, d\lambda = 0.$$

Indeed, by Cauchy's theorem it follows that

$$\begin{aligned}\int_\Gamma \frac{1}{\lambda} R(\lambda) A x_0 \, d\lambda &= \lim_{r\to\infty} \int_{\Gamma\cap\{|\lambda|\leq r\}} \frac{1}{\lambda} R(\lambda) A x_0 \, d\lambda \\ &= -\lim_{r\to\infty} \int_{C_r} \frac{1}{\lambda} R(\lambda) A x_0 \, d\lambda \\ &= 0,\end{aligned}$$

where C_r is a closed path shown in Figure 2.7.

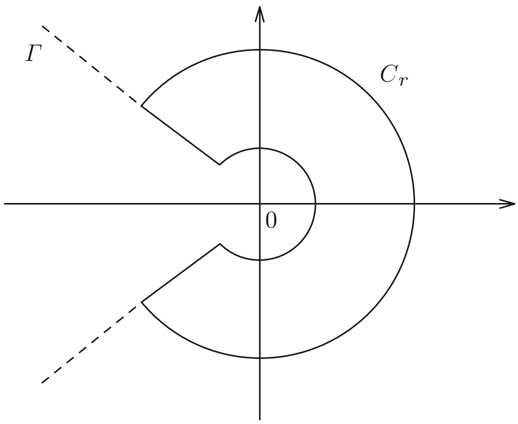

Fig. 2.7.

Summing up, we have proved that

$$U(z)x_0 \longrightarrow x_0 \quad \text{as } z \to 0,\ z \in \Delta_\omega^{2\varepsilon},$$

for each $x_0 \in \mathcal{D}(A)$.

Since the domain $\mathcal{D}(A)$ is dense in E and $\|U(z)\| \leq \widetilde{M}_0(\varepsilon)$ for all $z \in \Delta_\omega^{2\varepsilon}$, it follows that, for each $x \in E$,

$$U(z)x \longrightarrow x \quad \text{as } z \to 0,\ z \in \Delta_\omega^{2\varepsilon}.$$

The proof of Theorem 2.2 is now complete. □

Remark 2.1. Assume that the operator A satisfies a stronger condition than condition (2.1):

$$\|R(\lambda)\| \leq \frac{M_\varepsilon}{|\lambda|+1} \quad \text{for all } \lambda \in \Sigma_\omega^\varepsilon. \tag{2.15}$$

Then we have the estimates

$$\|U(z)\| \leq \widetilde{M_0}(\varepsilon) e^{-\delta \cdot \operatorname{Re} z} \qquad \text{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.16}$$

$$\|AU(z)\| \leq \frac{\widetilde{M_1}(\varepsilon)}{|z|} e^{-\delta \cdot \operatorname{Re} z} \quad \text{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.17}$$

with some positive constant δ.

Proof. Take a real number δ such that

$$0 < \delta < \frac{1}{M_\varepsilon}.$$

Then we have, by estimate (2.15),

$$\delta \left\|(A-\lambda I)^{-1}\right\| \leq \frac{\delta M_\varepsilon}{|\lambda|+1} \leq \delta M_\varepsilon < 1 \quad \text{for all } \lambda \in \Sigma_\omega^\varepsilon.$$

Hence it follows that the operator $(A+\delta I)-\lambda I$ has the inverse

$$((A+\delta I)-\lambda I)^{-1} = (I+\delta(A-\lambda I)^{-1})^{-1}(A-\lambda I)^{-1},$$

and

$$\begin{aligned}
\|((A+\delta I)-\lambda I)^{-1}\| &\leq \|(I+\delta(A-\lambda I)^{-1})^{-1}\| \cdot \|(A-\lambda I)^{-1}\| \\
&\leq \frac{M_\varepsilon}{|\lambda|+1} \frac{1}{1-\|\delta(A-\lambda I)^{-1}\|} \\
&\leq \frac{M_\varepsilon}{|\lambda|+1} \frac{1}{1-\delta M_\varepsilon} \\
&\leq \left(\frac{M_\varepsilon}{1-\delta M_\varepsilon}\right) \frac{1}{|\lambda|}.
\end{aligned}$$

This proves that the operator $A+\delta I$ satisfies condition (2.1), so that estimates (2.4) and (2.5) remain valid for the operator $A+\delta I$:

$$\|V(z)\| \leq \widetilde{M_0}(\varepsilon) \qquad \text{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.18}$$

$$\|(A+\delta I)V(z)\| \leq \frac{\widetilde{M_1}(\varepsilon)}{|z|} \quad \text{for all } z \in \Delta_\omega^{2\varepsilon}, \tag{2.19}$$

where

$$V(z) = -\frac{1}{2\pi i} \int_\Gamma e^{\lambda z} (A+\delta I-\lambda I)^{-1} \, d\lambda.$$

However, we have, by Cauchy's theorem,

$$\begin{aligned} V(z) &= -\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda z}(A+\delta I-\lambda I)^{-1}\,d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma+\delta} e^{\lambda z}(A+\delta I-\lambda I)^{-1}\,d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma} e^{\mu z}e^{\delta z}(A-\mu I)^{-1}\,d\mu = e^{\delta z}U(z) \\ &\quad \text{for all } z\in\Delta_{\omega}^{2\varepsilon}. \end{aligned} \tag{2.20}$$

In view of formula (2.16), the desired estimates (2.16) and (2.17) follow from estimates (2.18) and (2.19). □

2.1.2 Fractional Powers

Assume that the operator A satisfies a stronger condition than condition (2.1):

(1) The resolvent set of A contains the region Σ as in Figure 2.8.

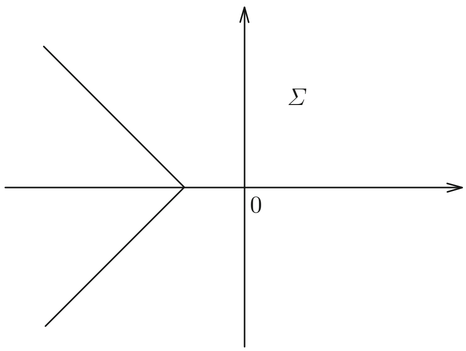

Fig. 2.8.

(2) There exists a positive constant M such that the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ satisfies the estimate

$$\|R(\lambda)\| \leq \frac{M}{(1+|\lambda|)} \quad \text{for all } \lambda\in\Sigma. \tag{2.21}$$

If $\alpha > 0$, we can define the fractional power $(-A)^{-\alpha}$ of $-A$ by the following formula:

$$(-A)^{-\alpha} = -\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\alpha}R(\lambda)\,d\lambda. \tag{2.22}$$

Here the path Γ runs in the set Σ from $\infty e^{-i\omega}$ to $\infty e^{i\omega}$, avoiding the positive real axis and the origin (see Figure 2.9), and for the function $(-\lambda)^{-\alpha} =$

$e^{-\alpha \log(-\lambda)}$, we choose the branch whose argument lies between $-\alpha\pi$ and $\alpha\pi$; it is analytic in the region obtained by omitting the positive real axis.

The integral (2.22) converges in the uniform operator topology of the Banach space $L(E,E)$ for all $\alpha > 0$, and thus defines a bounded linear operator on E.

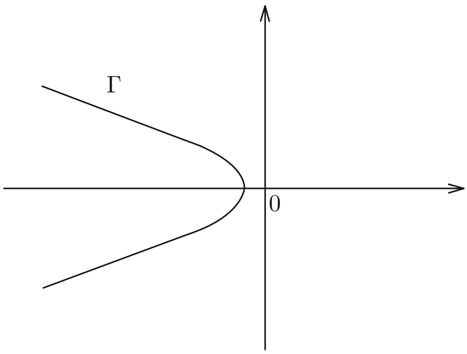

Fig. 2.9.

Some basic properties of the fractional power $(-A)^{-\alpha}$ are summarized in the following:

Proposition 2.3. *(i) We have, for all* α, $\beta > 0$,

$$(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}.$$

(ii) If α *is a positive integer* n*, then we have the formula*

$$(-A)^{-\alpha} = \left((-A)^{-1}\right)^n .$$

(iii) The fractional power $(-A)^{-\alpha}$ *is invertible for all* $\alpha > 0$.

If $0 < \alpha < 1$, we have the following useful formula for the fractional power $(-A)^{-\alpha}$:

Theorem 2.4. *We have, for all* $0 < \alpha < 1$,

$$(-A)^{-\alpha} = -\frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha} R(s) ds. \tag{2.23}$$

By Remark 2.1, we may assume that there exist positive constants M_0, M_1 and a such that

$$\|U(t)\| \leq M_0 e^{-at} \qquad \text{for all } t > 0.$$

$$\|AU(t)\| \leq M_1 e^{-at} \frac{1}{t} \quad \text{for all } t > 0.$$

Then we can prove still another useful formula for the fractional power $(-A)^{-\alpha}$ for all $0 < \alpha < 1$.

Theorem 2.5. *We have, for all* $0 < \alpha < 1$,

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} U(t)\, dt.$$

In view of part (iii) of Proposition 2.3, we can define the fractional power $(-A)^\alpha$ for all $\alpha > 0$ as follows:

$$(-A)^\alpha = \text{ the inverse of } (-A)^{-\alpha}, \quad \alpha > 0.$$

The next theorem states that the domain $\mathcal{D}((-A)^\alpha)$ of $(-A)^\alpha$ is bigger than the domain $\mathcal{D}(A)$ of A when $0 < \alpha < 1$.

Theorem 2.6. *We have, for all* $0 < \alpha < 1$,

$$\mathcal{D}(A) \subset \mathcal{D}\left((-A)^\alpha\right).$$

We can give an explicit formula for the fractional power $(-A)^\alpha$ $(0 < \alpha < 1)$ on the domain $\mathcal{D}(A)$:

Theorem 2.7. *Let* $0 < \alpha < 1$. *Then we have, for any* $x \in \mathcal{D}(A)$,

$$(-A)^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{\alpha-1} R(s) A x\, ds.$$

2.1.3 The Semilinear Cauchy Problem

This subsection is devoted to the semigroup approach to a class of initial-boundary value problems for *semilinear* parabolic differential equations. By making good use of fractional powers of analytic semigroups, we formulate a local existence and uniqueness theorem for semilinear initial-boundary value problems (Theorem 2.8). Our semigroup approach can be traced back to the pioneering work of Fujita–Kato [FK] for the Navier–Stokes equation in fluid dynamics.

Assume that the operator A satisfies condition (2.21). Then we can define the fractional power $(-A)^\alpha$ for all $0 < \alpha < 1$ (see Subsection 2.1.2). The operator $(-A)^\alpha$ is a closed linear, invertible operator with domain $\mathcal{D}((-A)^\alpha) \supset \mathcal{D}(A)$. We let

$$E_\alpha = \begin{array}{l}\text{the space } \mathcal{D}((-A)^\alpha) \text{ endowed with the graph norm } \|\cdot\|_\alpha \\ \text{of } (-A)^\alpha,\end{array}$$

where

$$\|x\|_\alpha = \left(\|x\|^2 + \|(-A)^\alpha x\|^2\right)^{1/2}, \quad x \in \mathcal{D}((-A)^\alpha).$$

Then we have the following three assertions:

(i) The space E_α is a Banach space.

(ii) The graph norm $\|x\|_\alpha$ is equivalent to the norm $\|(-A)^\alpha x\|$.
(iii) If $0 < \alpha < \beta < 1$, then we have $E_\beta \subset E_\alpha$ with continuous injection.

Now we consider the following *semilinear* Cauchy problem:

$$\begin{cases} \frac{du}{dt} = Au(t) + f(t, u(t)), & t_0 < t < t_1, \\ u(t_0) = x_0. \end{cases} \tag{2.24}$$

Here $f(t, x)$ is a function defined on an open subset U of $[0, \infty) \times E_\alpha$ $(0 < \alpha < 1)$, taking values in E. We assume that $f(t, x)$ is locally Hölder continuous in t and locally Lipschitz continuous in x. That is, for each point (t, x) of U there exist a neighborhood $V \subset U$, constants $L = L(t, x, V) > 0$ and $0 < \gamma \leq 1$ such that

$$\|f(s_1, y_1) - f(s_2, y_2)\| \leq L \left(|s_1 - s_2|^\gamma + \|y_1 - y_2\|_\alpha\right), \\ (s_1, y_1), (s_2, y_2) \in V.$$

A function $u(t) : [t_0, t_1) \to E$ is called a *solution* of problem (2.24) if it satisfies the following three conditions:

(1) $u(t) \in C([t_0, t_1); E) \cap C^1((t_0, t_1); E)$ and $u(t_0) = x_0$.
(2) $u(t) \in \mathcal{D}(A)$ and $(t, u(t)) \in U$ for all $t_0 < t < t_1$.
(3) $\frac{du}{dt} = Au(t) + f(t, u(t))$ for all $t_0 < t < t_1$.

Here $C([t_0, t_1); E)$ denotes the space of continuous functions on $[t_0, t_1)$ taking values in E, and $C^1((t_0, t_1); E)$ denotes the space of continuously differentiable functions on (t_0, t_1) taking values in E, respectively.

Our main result is the following local existence and uniqueness theorem for problem (2.24):

Theorem 2.8. *Let $f(t, x)$ be a function defined on an open subset U of $[0, \infty) \times E_\alpha$ $(0 < \alpha < 1)$, taking values in E. Assume that $f(t, x)$ is locally Hölder continuous in t and locally Lipschitz continuous in x. Then, for every $(t_0, x_0) \in U$, problem (2.24) has a unique local solution $u(t) \in C([t_0, t_1]; E) \cap C^1((t_0, t_1); E)$ where $t_1 = t_1(t_0, x_0) > t_0$.*

For a proof of Theorem 2.8, the reader is referred to [He, Theorem 3.3.3], Pazy [Pa, Chapter 6, Theorem 3.1] and also Taira [Ta4, Theorem 1.18].

2.2 Markov Processes and Feller Semigroups

This section provides a brief description of basic definitions and results about Markov processes and a class of semigroups (Feller semigroups) associated with Markov processes. In Subsection 2.2.6 we prove various generation theorems of Feller semigroups by using the Hille–Yosida theory of semigroups (Theorems 2.16 and 2.18) which form a functional analytic background for the proof of Theorem 1.4. The results discussed here are adapted from

Blumenthal–Getoor [BG], Dynkin [Dy2], Lamperti [La], Revuz–Yor [RY] and also Taira [Ta2, Chapter 9]. The semigroup approach to Markov processes can be traced back to the pioneering work of Feller [Fe1] and [Fe2] in early 1950s (cf. [BCP], [SU], [Ta3]).

2.2.1 Markov Processes

In 1828 the English botanist R. Brown observed that pollen grains suspended in water move chaotically, incessantly changing their direction of motion. The physical explanation of this phenomenon is that a single grain suffers innumerable collisions with the randomly moving molecules of the surrounding water. A mathematical theory for Brownian motion was put forward by A. Einstein in 1905 (cf. [Ei]). Let $p(t, x, y)$ be the probability density function that a one-dimensional Brownian particle starting at position x will be found at position y at time t. Einstein derived the following formula from statistical mechanical considerations:

$$p(t, x, y) = \frac{1}{\sqrt{2\pi Dt}} \exp\left[-\frac{(y-x)^2}{2Dt}\right].$$

Here D is a positive constant determined by the radius of the particle, the interaction of the particle with surrounding molecules, temperature and the Boltzmann constant. This gives an accurate method of measuring Avogadro's number by observing particles. Einstein's theory was experimentally tested by J. Perrin between 1906 and 1909.

Brownian motion was put on a firm mathematical foundation for the first time by N. Wiener in 1923 ([Wi]). Let Ω be the space of continuous functions $\omega : [0, \infty) \mapsto \mathbf{R}$ with coordinates $x_t(\omega) = \omega(t)$ and let $\mathcal{F}$ be the smallest σ-algebra in Ω which contains all sets of the form $\{\omega \in \Omega : a \leq x_t(\omega) < b\}$, $t \geq 0$, $a < b$. Wiener constructed probability measures P_x, $x \in \mathbf{R}$, on $\mathcal{F}$ for which the following formula holds:

$$\begin{aligned} P_x\{\omega \in \Omega : a_1 \leq x_{t_1}(\omega) < b_1, a_2 \leq x_{t_2}(\omega) < b_2, \ldots, a_n \leq x_{t_n}(\omega) < b_n\} \\ = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} p(t_1, x, y_1) p(t_2 - t_1, y_1, y_2) \ldots \\ p(t_n - t_{n-1}, y_{n-1}, y_n)\, dy_1\, dy_2 \ldots dy_n, \\ 0 < t_1 < t_2 < \ldots < t_n < \infty. \end{aligned}$$

This formula expresses the "starting afresh" property of Brownian motion that if a Brownian particle reaches a position, then it behaves subsequently as though that position had been its initial position. The measure P_x is called the *Wiener measure* starting at x.

P. Lévy found another construction of Brownian motion, and gave a profound description of qualitative properties of the individual Brownian path in his book: Processus stochastiques et mouvement brownien (1948).

Markov processes are an abstraction of the idea of Brownian motion. Let K be a locally compact, separable metric space and let $\mathcal{B}$ be the σ-algebra of all Borel sets in K, that is, the smallest σ-algebra containing all open sets in K. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X(\omega)$ defined on Ω taking values in K is called a *random variable* if it satisfies the condition

$$X^{-1}(E) = \{\omega \in \Omega : X(\omega) \in E\} \in \mathcal{F} \quad \text{for all } E \in \mathcal{B}.$$

We express this by saying that X is $\mathcal{F}/\mathcal{B}$-measurable. A family $\{x_t\}_{t \geq 0}$ of random variables is called a *stochastic process*, and it may be thought of as the motion in time of a physical particle. The space K is called the *state space* and Ω the *sample space*. For a fixed $\omega \in \Omega$, the function $x_t(\omega)$, $t \geq 0$, defines in the state space K a *trajectory* or *path* of the process corresponding to the sample point ω.

In this generality the notion of a stochastic process is of course not so interesting. The most important class of stochastic processes is the class of Markov processes which is characterized by the Markov property. Intuitively, this is the principle of the lack of any "memory" in the system. More precisely, (temporally homogeneous) *Markov property* is that the prediction of subsequent motion of a particle, knowing its position at time t, depends neither on the value of t nor on what has been observed during the time interval $[0, t)$; that is, a particle "starts afresh".

Now we introduce a class of Markov processes which we will deal with in this book (cf. [Dy2], [BG], [RY]).

Assume that we are given the following:

(1) A locally compact, separable metric space K and the σ-algebra $\mathcal{B}$ of all Borel sets in K. A point ∂ is adjoined to K as the point at infinity if K is not compact, and as an isolated point if K is compact (see Figure 2.10). We let

$$\begin{aligned} K_\partial &= K \cup \{\partial\}, \\ \mathcal{B}_\partial &= \text{the } \sigma\text{-algebra in } K_\partial \text{ generated by } \mathcal{B}. \end{aligned}$$

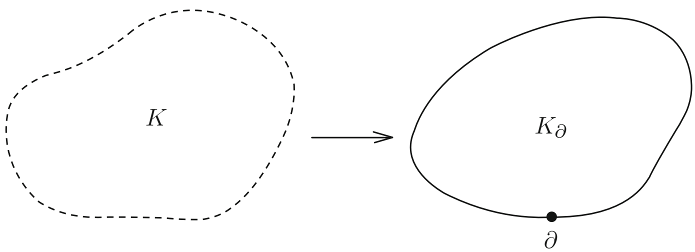

Fig. 2.10.

(2) The space Ω of all mappings $\omega : [0, \infty] \to K_\partial$ such that $\omega(\infty) = \partial$ and that if $\omega(t) = \partial$ then $\omega(s) = \partial$ for all $s \geq t$. Let ω_∂ be the constant map $\omega_\partial(t) = \partial$ for all $t \in [0, \infty]$.

(3) For each $t \in [0, \infty]$, the coordinate map x_t defined by $x_t(\omega) = \omega(t)$, $\omega \in \Omega$.
(4) For each $t \in [0, \infty]$, a mapping $\varphi_t : \Omega \to \Omega$ defined by $(\varphi_t \omega)(s) = \omega(t+s)$, $\omega \in \Omega$. Note that $\varphi_\infty \omega = \omega_\partial$ and $x_t \circ \varphi_s = x_{t+s}$ for all t, $s \in [0, \infty]$.
(5) A σ-algebra $\mathcal{F}$ in Ω and an increasing family $\{\mathcal{F}_t\}_{0 \leq t \leq \infty}$ of sub-σ-algebras of $\mathcal{F}$.
(6) For each $x \in K_\partial$, a probability measure P_x on $(\Omega, \mathcal{F})$.

We say that these elements define a (temporally homogeneous) *Markov process* $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ if the following four conditions are satisfied:

(i) For each $0 \leq t < \infty$, the function x_t is $\mathcal{F}_t/\mathcal{B}_\partial$-measurable, that is,

$$\{x_t \in E\} = \{\omega \in \Omega : x_t(\omega) \in E\} \in \mathcal{F}_t \quad \text{for all } E \in \mathcal{B}_\partial.$$

(ii) For all $0 \leq t < \infty$ and $E \in \mathcal{B}$, the function

$$p_t(x, E) = P_x\{x_t \in E\} \tag{2.25}$$

is a Borel measurable function of $x \in K$.
(iii) $P_x\{\omega \in \Omega : x_0(\omega) = x\} = 1$ for each $x \in K_\partial$.
(iv) For all t, $h \in [0, \infty]$, $x \in K_\partial$ and $E \in \mathcal{B}_\partial$, we have the formula

$$P_x\{x_{t+h} \in E \mid \mathcal{F}_t\} = p_h(x_t, E) \quad \text{a. e.,}$$

or equivalently,

$$P_x(A \cap \{x_{t+h} \in E\}) = \int_A p_h(x_t(\omega), E)\, dP_x(\omega) \quad \text{for all } A \in \mathcal{F}_t.$$

Here is an intuitive way of thinking about the above definition of a Markov process. The sub-σ-algebra $\mathcal{F}_t$ may be interpreted as the collection of events which are observed during the time interval $[0, t]$. The value $P_x(A)$, $A \in \mathcal{F}$, may be interpreted as the probability of the event A under the condition that a particle starts at position x; hence the value $p_t(x, E)$ expresses the transition probability that a particle starting at position x will be found in the set E at time t (see Figure 2.11). The function $p_t(x, \cdot)$ is called the *transition function* of the process $\mathcal{X}$. The transition function $p_t(x, \cdot)$ specifies the probability structure of the process. The intuitive meaning of the crucial condition (iv) is that the future behavior of a particle, knowing its history up to time t, is the same as the behavior of a particle starting at $x_t(\omega)$, that is, a particle starts afresh. A particle moves in the space K until it "dies" at the time when it reaches the point ∂; hence the point ∂ is called the *terminal point.*

With this interpretation in mind, we let

$$\zeta(\omega) = \inf\{t \in [0, \infty] : x_t(\omega) = \partial\}.$$

The random variable ζ is called the *lifetime* of the process $\mathcal{X}$.

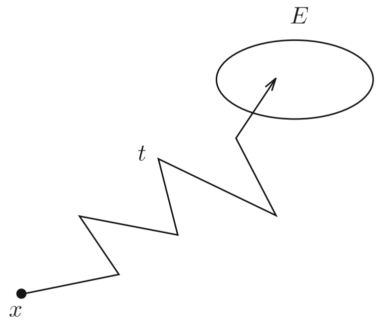

Fig. 2.11.

2.2.2 Markov Transition Functions

In the first works devoted to Markov processes, the most fundamental was A. N. Kolmogorov's work ([Ko]) where the general concept of a Markov transition function was introduced for the first time and an analytic method of describing Markov transition functions was proposed. From the point of view of analysis, the transition function is something more convenient than the Markov process itself. In fact, it can be shown that the transition functions of Markov processes generate solutions of certain parabolic partial differential equations such as the classical diffusion equation; and, conversely, these differential equations can be used to construct and study the transition functions and the Markov processes themselves.

In the 1950s, the theory of Markov processes entered a new period of intensive development. We can associate with each transition function in a natural way a family of bounded linear operators acting on the space of continuous functions on the state space, and the Markov property implies that this family forms a semigroup. The Hille–Yosida theory of semigroups in functional analysis made possible further progress in the study of Markov processes, as will be shown in Subsection 2.2.5.

Our first job is thus to give the precise definition of a transition function adapted to the theory of semigroups:

Definition 2.1. Let (K, ρ) be a locally compact, separable metric space and let $\mathcal{B}$ be the σ-algebra of all Borel sets in K. A function $p_t(x, E)$, defined for all $t \geq 0$, $x \in K$ and $E \in \mathcal{B}$, is called a (temporally homogeneous) *Markov transition function* on K if it satisfies the following four conditions:

(a) $p_t(x, \cdot)$ is a non-negative measure on $\mathcal{B}$ and $p_t(x, K) \leq 1$ for all $t \geq 0$ and $x \in K$.
(b) $p_t(\cdot, E)$ is a Borel measurable function for all $t \geq 0$ and $E \in \mathcal{B}$.
(c) $p_0(x, \{x\}) = 1$ for all $x \in K$.

(d) (The Chapman–Kolmogorov equation) For all t, $s \geq 0$, $x \in K$ and $E \in \mathcal{B}$, we have the equation

$$p_{t+s}(x, E) = \int_K p_t(x, dy) p_s(y, E). \tag{2.26}$$

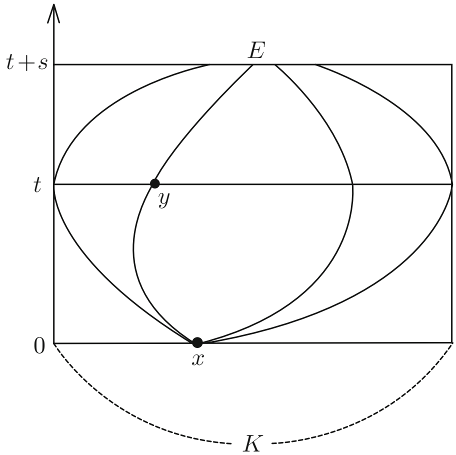

Fig. 2.12.

Here is an intuitive way of thinking about the above definition of a Markov transition function. The value $p_t(x, E)$ expresses the transition probability that a physical particle starting at position x will be found in the set E at time t. The Chapman–Kolmogorov equation (2.26) expresses the idea that a transition from the position x to the set E in time $t + s$ is composed of a transition from x to some position y in time t, followed by a transition from y to the set E in the remaining time s; the latter transition has probability $p_s(y, E)$ which depends only on y (see Figure 2.12). Thus a particle "starts afresh"; this property is called the *Markov property*.

The Chapman–Kolmogorov equation (2.26) asserts that $p_t(x, K)$ is monotonically increasing as $t \downarrow 0$, so that the limit

$$p_{+0}(x, K) = \lim_{t \downarrow 0} p_t(x, K)$$

exists.

A transition function $p_t(x, \cdot)$ is said to be *normal* if it satisfies the condition

$$p_{+0}(x, K) = 1 \quad \text{for all } x \in K.$$

The next theorem, due to Dynkin [Dy1, Chapter 4, Section 2], justifies the definition of a transition function, and hence it will be fundamental for our further study of Markov processes:

Theorem 2.9. *For every Markov process, the function $p_t(x, \cdot)$, defined by formula (2.25), is a Markov transition function. Conversely, every normal Markov transition function corresponds to some Markov process.*

Here are some important examples of normal transition functions on the line $\mathbf{R} = (-\infty, \infty)$:

Example 2.1 (Uniform motion). If $t \geq 0$, $x \in \mathbf{R}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = \chi_E(x + vt),$$

where v is a constant, and $\chi_E(y) = 1$ if $y \in E$ and $= 0$ if $y \notin E$.

This process, starting at x, moves deterministically with constant velocity v.

Example 2.2 (Poisson process). If $t \geq 0$, $x \in \mathbf{R}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \chi_E(x + n),$$

where λ is a positive constant.

This process, starting at x, advances one unit by jumps, and the probability of n jumps during the time 0 and t is equal to $e^{-\lambda t}(\lambda t)^n/n!$.

Example 2.3 (Brownian motion). If $t > 0$, $x \in \mathbf{R}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = \frac{1}{\sqrt{2\pi t}} \int_E \exp\left[-\frac{(y-x)^2}{2t}\right] dy,$$

and

$$p_0(x, E) = \chi_E(x).$$

This is a mathematical model of one-dimensional Brownian motion. Its character is quite different from that of the Poisson process; the transition function $p_t(x, E)$ satisfies the condition

$$p_t(x, [x - \varepsilon, x + \varepsilon]) = 1 - o(t) \quad \text{as } t \downarrow 0,$$

for all $\varepsilon > 0$ and $x \in \mathbf{R}$. This means that the process never stands still, as does the Poisson process. Indeed, this process changes state not by jumps but by *continuous* motion. A Markov process with this property is called a *diffusion process*.

Example 2.4 (Brownian motion with constant drift). If $t > 0$, $x \in \mathbf{R}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = \frac{1}{\sqrt{2\pi t}} \int_E \exp\left[-\frac{(y - mt - x)^2}{2t}\right] dy,$$

and

$$p_0(x, E) = \chi_E(x),$$

where m is a constant.

This represents Brownian motion with a constant drift of magnitude m superimposed; the process can be represented as $\{x_t + mt\}$, where $\{x_t\}$ is Brownian motion on $\mathbf{R}$.

Example 2.5 (Cauchy process). If $t > 0$, $x \in \mathbf{R}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = \frac{1}{\pi} \int_E \frac{t}{t^2 + (y - x)^2} \, dy,$$

and

$$p_0(x, E) = \chi_E(x).$$

This process can be thought of as the "trace" on the real line of trajectories of two-dimensional Brownian motion, and it moves by jumps (see [Kn, Lemma 2.12]). More precisely, if $B_1(t)$ and $B_2(t)$ are two independent Brownian motions and if T is the first passage time of $B_1(t)$ to x, then $B_2(T)$ has the Cauchy density

$$\frac{1}{\pi} \frac{|x|}{x^2 + y^2}, \quad -\infty < y < \infty.$$

Here are two more examples of diffusion processes on the half line $\overline{\mathbf{R}^+} = [0, \infty)$ in which we must take account of the effect of the boundary point 0:

Example 2.6 (Reflecting barrier Brownian motion). If $t > 0$, $x \in \overline{\mathbf{R}^+}$ and $E \in \mathcal{B}$, we let

$$p_t(x, E) = \frac{1}{\sqrt{2\pi t}} \left(\int_E \exp\left[-\frac{(y - x)^2}{2t} \right] dy + \int_E \exp\left[-\frac{(y + x)^2}{2t} \right] dy \right),$$

and

$$p_0(x, E) = \chi_E(x).$$

This represents Brownian motion with a reflecting barrier at $x = 0$; the process may be represented as $\{|x_t|\}$, where $\{x_t\}$ is Brownian motion on $\mathbf{R}$. Indeed, since $\{|x_t|\}$ goes from x to y if $\{x_t\}$ goes from x to $\pm y$ due to the symmetry of the transition function in Example 2.3 about $x = 0$, it follows that

$$\begin{aligned} p_t(x, E) &= P_x\{|x_t| \in E\} \\ &= \frac{1}{\sqrt{2\pi t}} \left(\int_E \exp\left[-\frac{(y - x)^2}{2t} \right] dy + \int_E \exp\left[-\frac{(y + x)^2}{2t} \right] dy \right). \end{aligned}$$

Example 2.7 (Sticking barrier Brownian motion). If $t > 0$, $x \in \overline{\mathbf{R}^+}$ and $E \in \mathcal{B}$, we let

$$\begin{aligned} p_t(x, E) = \frac{1}{\sqrt{2\pi t}} &\left(\int_E \exp\left[-\frac{(y - x)^2}{2t} \right] dy - \int_E \exp\left[-\frac{(y + x)^2}{2t} \right] dy \right) \\ &+ \left(1 - \frac{1}{\sqrt{2\pi t}} \int_{-x}^{x} \exp\left[-\frac{z^2}{2t} \right] dz \right) \chi_E(0), \end{aligned}$$

and

$$p_0(x, E) = \chi_E(x).$$

This represents Brownian motion with a sticking barrier at $x = 0$. When a Brownian particle reaches the boundary point 0 for the first time, instead of reflecting it sticks there forever; in this case the state 0 is called a *trap*.

2.2.3 Path Functions of Markov Processes

It is naturally interesting and important to ask the following problem:

Problem. Given a Markov transition function $p_t(x, \cdot)$, under which conditions on $p_t(x, \cdot)$ does there exist a Markov process with transition function $p_t(x, \cdot)$ whose paths are almost surely continuous?

A Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ is said to be *right continuous* provided that we have, for each $x \in K$,

$$\begin{aligned}&P_x\{\omega \in \Omega : \text{the mapping } t \mapsto x_t(\omega) \text{ is a right continuous function} \\ &\quad \text{from } [0, \infty) \text{ into } K_\partial\} = 1.\end{aligned}$$

Furthermore, we say that $\mathcal{X}$ is *continuous* provided that we have, for each $x \in K$,

$$\begin{aligned}&P_x\{\omega \in \Omega : \text{the mapping } t \mapsto x_t(\omega) \text{ is a continuous function} \\ &\quad \text{from } [0, \zeta(\omega)) \text{ into } K_\partial\} = 1,\end{aligned}$$

where ζ is the lifetime of the process $\mathcal{X}$.

Now we give some useful criteria for path continuity in terms of Markov transition functions (see Dynkin [Dy1, Chapter 6], [Dy2, Chapter 3, Section 2]):

Theorem 2.10. *Let (K, ρ) be a locally compact, separable metric space and let $p_t(x, \cdot)$ be a normal Markov transition function on K.*
(i) Assume that the following two conditions are satisfied:

(L) For each $s > 0$ and each compact $E \subset K$, we have the condition

$$\lim_{x \to \partial} \sup_{0 \le t \le s} p_t(x, E) = 0.$$

(M) For each $\varepsilon > 0$ and each compact $E \subset K$, we have the condition

$$\lim_{t \downarrow 0} \sup_{x \in E} p_t(x, K \setminus U_\varepsilon(x)) = 0,$$

where $U_\varepsilon(x) = \{y \in K : \rho(y, x) < \varepsilon\}$ is an ε-neighborhood of x.

Then there exists a Markov process $\mathcal{X}$ with transition function $p_t(x,\cdot)$ whose paths are right continuous on $[0,\infty)$ and have left-hand limits on $[0,\zeta)$ almost surely.
(ii) Assume that condition (L) and the following condition (N) (replacing condition (M)) are satisfied:

(N) For each $\varepsilon > 0$ and each compact $E \subset K$, we have the condition

$$\lim_{t\downarrow 0} \frac{1}{t} \sup_{x\in E} p_t(x, K \setminus U_\varepsilon(x)) = 0,$$

or equivalently

$$\sup_{x\in E} p_t(x, K \setminus U_\varepsilon(x)) = o(t) \quad \text{as } t \downarrow 0.$$

Then there exists a Markov process $\mathcal{X}$ with transition function $p_t(x,\cdot)$ whose paths are almost surely continuous on $[0,\zeta)$.

Remark 2.2. It is known (see Dynkin [Dy1, Lemma 6.2]) that if the paths of a Markov process are right continuous, then the transition function $p_t(x,\cdot)$ satisfies the condition

$$\lim_{t\downarrow 0} p_t(x, U_\varepsilon(x)) = 1 \quad \text{for all } x \in K.$$

2.2.4 Strong Markov Processes and Transition Functions

A Markov process is called a *strong Markov process* if the "starting afresh" property holds not only for every fixed moment but also for suitable random times.

We shall formulate precisely this "strong" Markov property. Let $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ be a Markov process. A mapping $\tau\colon \Omega \to [0,\infty]$ is called a *stopping time* or *Markov time* with respect to $\{\mathcal{F}_t\}$ if it satisfies the condition

$$\{\tau \le t\} = \{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}_t \quad \text{for all } t \in [0,\infty).$$

Intuitively, this means that the events $\{\tau \le t\}$ depend on the process only up to time t, but not on the "future" after time t. It should be noticed that any non-negative constant mapping is a stopping time.

If τ is a stopping time with respect to $\{\mathcal{F}_t\}$, we let

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \quad \text{for all } t \in [0,\infty)\}.$$

Intuitively, we may think of $\mathcal{F}_\tau$ as the "past" up to the random time τ. It is easy to verify that $\mathcal{F}_\tau$ is a σ-algebra. If $\tau \equiv t_0$ for some constant $t_0 \ge 0$, then $\mathcal{F}_\tau$ reduces to $\mathcal{F}_{t_0}$.

For each $t \in [0,\infty]$, we define a mapping

$$\Phi_t\colon [0,t] \times \Omega \longrightarrow K_\partial$$

by the formula

$$\Phi_t(s,\omega) = x_s(\omega).$$

A Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ is said to be *progressively measurable* with respect to $\{\mathcal{F}_t\}$ if the mapping Φ_t is $\mathcal{B}_{[0,t]} \times \mathcal{F}_t/\mathcal{B}_\partial$-measurable for each $t \in [0,\infty]$, that is, if we have the condition

$$\Phi_t^{-1}(E) = \{\Phi_t \in E\} \in \mathcal{B}_{[0,t]} \times \mathcal{F}_t \quad \text{for all } E \in \mathcal{B}_\partial.$$

Here $\mathcal{B}_{[0,t]}$ is the σ-algebra of all Borel sets in the interval $[0,t]$ and $\mathcal{B}_\partial$ is the σ-algebra in K_∂ generated by $\mathcal{B}$. It should be noticed that if $\mathcal{X}$ is progressively measurable and if τ is a stopping time, then the mapping $x_\tau\colon \omega \mapsto x_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau/\mathcal{B}_\partial$-measurable.

The next definition expresses the idea of "starting afresh" at random times:

Definition 2.2. We say that a progressively measurable Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ has the *strong Markov property* with respect to $\{\mathcal{F}_t\}$ if the following condition is satisfied: For all $h \geq 0$, $x \in K_\partial$, $E \in \mathcal{B}_\partial$ and all stopping times τ, we have the formula

$$P_x\{x_{\tau+h} \in E \mid \mathcal{F}_\tau\} = p_h(x_\tau, E),$$

or equivalently,

$$P_x(A \cap \{x_{\tau+h} \in E\}) = \int_A p_h(x_{\tau(\omega)}(\omega), E)\, dP_x(\omega) \quad \text{for all } A \in \mathcal{F}_\tau.$$

We shall state a simple criterion for the strong Markov property in terms of transition functions.

Let (K,ρ) be a locally compact, separable metric space. We add a point ∂ to the metric space K as the point at infinity if K is not compact, and as an isolated point if K is compact; so the space $K_\partial = K \cup \{\partial\}$ is compact (see Figure 2.10). Let $C(K)$ be the space of real-valued, bounded continuous functions $f(x)$ on K; the space $C(K)$ is a Banach space with the supremum norm

$$\|f\|_\infty = \sup_{x\in K} |f(x)|.$$

We say that a function $f \in C(K)$ converges to zero as $x \to \partial$ if, for each $\varepsilon > 0$, there exists a compact subset E of K such that

$$|f(x)| < \varepsilon \quad \text{for all } x \in K \setminus E,$$

and we then write $\lim_{x\to\partial} f(x) = 0$. We let

$$C_0(K) = \left\{ f \in C(K) : \lim_{x\to\partial} f(x) = 0 \right\}.$$

The space $C_0(K)$ is a closed subspace of $C(K)$; hence it is a Banach space. Note that $C_0(K)$ may be identified with $C(K)$ if K is compact.

Now we introduce a useful convention as follows:

Any real-valued function $f(x)$ on K is extended to the space $K_\partial = K \cup \{\partial\}$ by setting $f(\partial) = 0$.

From this point of view, the space $C_0(K)$ is identified with the subspace of $C(K_\partial)$ which consists of all functions $f(x)$ satisfying the condition $f(\partial) = 0$:

$$C_0(K) = \{f \in C(K_\partial) : f(\partial) = 0\}.$$

Furthermore, we can extend a Markov transition function $p_t(x, \cdot)$ on K to a Markov transition function $p'_t(x, \cdot)$ on K_∂ by the formulas:

$$\begin{cases} p'_t(x, E) = p_t(x, E) & \text{for all } x \in K \text{ and } E \in \mathcal{B}, \\ p'_t(x, \{\partial\}) = 1 - p_t(x, K) & \text{for all } x \in K, \\ p'_t(\partial, K) = 0, \ p'_t(\partial, \{\partial\}) = 1. \end{cases}$$

Intuitively this means that a Markovian particle moves in the space K until it "dies" at the time when it reaches the point ∂; hence the point ∂ is called the *terminal point*.

Now we introduce some conditions on the measures $p_t(x, \cdot)$ related to continuity in $x \in K$, for fixed $t \geq 0$:

Definition 2.3. (i) A Markov transition function $p_t(x, \cdot)$ is called a *Feller function* if the function

$$T_t f(x) = \int_K p_t(x, dy) f(y)$$

is a continuous function of $x \in K$ whenever f is in $C(K)$, that is, if we have the condition

$$f \in C(K) \implies T_t f \in C(K).$$

(ii) We say that $p_t(x, \cdot)$ is a C_0-*function* if the space $C_0(K)$ is an invariant subspace of $C(K)$ for the operators T_t:

$$f \in C_0(K) \implies T_t f \in C_0(K).$$

Remark 2.3. The Feller property is equivalent to saying that the measures $p_t(x, \cdot)$ depend continuously on $x \in K$ in the usual weak topology, for every fixed $t \geq 0$.

The next theorem gives a useful criterion for the strong Markov property (see [Dy1, Theorem 5.10]):

Theorem 2.11. *If the transition function $p_t(x, \cdot)$ of a right continuous Markov process $\mathcal{X}$ has the C_0-property, then $\mathcal{X}$ is a strong Markov process.*

Furthermore, we state a simple criterion for the strong Markov property in terms of Markov transition functions. To do this, we introduce the following definition:

Definition 2.4. A Markov transition function $p_t(x, \cdot)$ on K is said to be *uniformly stochastically continuous* on K if the following condition is satisfied: For each $\varepsilon > 0$ and each compact $E \subset K$, we have the condition

$$\lim_{t \downarrow 0} \sup_{x \in E} \left[1 - p_t(x, U_\varepsilon(x))\right] = 0, \tag{2.27}$$

where $U_\varepsilon(x) = \{y \in K : \rho(y, x) < \varepsilon\}$ is an ε-neighborhood of x.

It should be emphasized that every uniformly stochastically continuous transition function is normal and satisfies condition (M) in Theorem 2.10. By combining part (i) of Theorem 2.10 and Theorem 2.11, we obtain the following result (see [Dy1, Theorem 6.3]):

Theorem 2.12. *Assume that a uniformly stochastically continuous, C_0-transition function $p_t(x, \cdot)$ satisfies condition (L). Then it is the transition function of some strong Markov process $\mathcal{X}$ whose paths are right continuous and have no discontinuities other than jumps.*

A continuous strong Markov process is called a *diffusion process*.

The next theorem states a sufficient condition for the existence of a diffusion process with a prescribed Markov transition function:

Theorem 2.13. *Assume that a uniformly stochastically continuous, C_0-transition function $p_t(x, \cdot)$ satisfies conditions (L) and (N). Then it is the transition function of some diffusion process $\mathcal{X}$.*

This is an immediate consequence of part (ii) of Theorem 2.10 and Theorem 2.12.

2.2.5 Markov Transition Functions and Feller Semigroups

The Feller or C_0-property deals with continuity of a Markov transition function $p_t(x, E)$ in x, and does not, by itself, have no concern with continuity in t. We give a necessary and sufficient condition on $p_t(x, E)$ in order that its associated operators $\{T_t\}_{t \geq 0}$, defined by the formula

$$T_t f(x) = \int_K p_t(x, dy) f(y), \quad f \in C_0(K), \tag{2.28}$$

is *strongly continuous* in t on the space $C_0(K)$:

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\|_\infty = 0, \quad f \in C_0(K). \tag{2.29}$$

Then we have the following (cf. [Ta2, Theorem 9.2.3]):

Theorem 2.14. *Let $p_t(x,\cdot)$ be a C_0-transition function on K. Then the associated operators $\{T_t\}_{t\geq 0}$, defined by formula (2.28) is strongly continuous in t on $C_0(K)$ if and only if $p_t(x,\cdot)$ is uniformly stochastically continuous on K and satisfies condition (L).*

Proof. (i) The "if" part: Since continuous functions with compact support are dense in $C_0(K)$, it suffices to prove the strong continuity of $\{T_t\}$ at $t=0$:

$$\lim_{t\downarrow 0} \|T_t f - f\|_\infty = 0 \tag{2.30}$$

for all such functions f.

For any compact subset E of K containing the support $\operatorname{supp} f$ of f, we have the inequality

$$\begin{aligned}\|T_t f - f\|_\infty &\leq \sup_{x\in E} |T_t f(x) - f(x)| + \sup_{x\in K\setminus E} |T_t f(x)| \\ &\leq \sup_{x\in E} |T_t f(x) - f(x)| + \|f\|_\infty \cdot \sup_{x\in K\setminus E} p_t(x, \operatorname{supp} f).\end{aligned} \tag{2.31}$$

However, condition (L) implies that, for each $\varepsilon > 0$, we can find a compact subset E of K such that, for all sufficiently small $t>0$,

$$\sup_{x\in K\setminus E} p_t(x, \operatorname{supp} f) < \varepsilon. \tag{2.32}$$

On the other hand, we have, for each $\delta > 0$,

$$\begin{aligned}T_t f(x) - f(x) &= \int_{U_\delta(x)} p_t(x,dy)(f(y)-f(x)) \\ &\quad + \int_{K\setminus U_\delta(x)} p_t(x,dy)(f(y)-f(x)) - f(x)(1-p_t(x,K)),\end{aligned}$$

and hence

$$\begin{aligned}&\sup_{x\in E} |T_t f(x) - f(x)| \\ &\leq \sup_{\rho(x,y)<\delta} |f(y)-f(x)| + 3\|f\|_\infty \cdot \sup_{x\in E} \left[1 - p_t(x, U_\delta(x))\right].\end{aligned}$$

Since the function $f(x)$ is uniformly continuous, we can choose a positive constant δ such that

$$\sup_{\rho(x,y)<\delta} |f(y)-f(x)| < \varepsilon.$$

Furthermore, it follows from condition (2.27) with $\varepsilon := \delta$ (the uniform stochastic continuity of $p_t(x,\cdot)$) that, for all sufficiently small $t>0$,

$$\sup_{x \in E} [1 - p_t(x, U_\delta(x))] < \varepsilon.$$

Hence we have, for all sufficiently small $t > 0$,

$$\sup_{x \in E} |T_t f(x) - f(x)| < \varepsilon(1 + 3\|f\|_\infty). \tag{2.33}$$

Therefore, by carrying inequalities (2.32) and (2.33) into inequality (2.31) we obtain that, for all sufficiently small $t > 0$,

$$\|T_t f - f\|_\infty < \varepsilon(1 + 4\|f\|_\infty).$$

This proves the desired formula (2.30), that is, the strong continuity (2.29) of $\{T_t\}$.

(ii) The "only if" part: For any $x \in K$ and $\varepsilon > 0$, we define a continuous function $f_x(y)$ by the formula (see Figure 2.13)

$$f_x(y) = \begin{cases} 1 - \frac{1}{\varepsilon}\rho(x, y) & \text{if } \rho(x, y) \leq \varepsilon, \\ 0 & \text{if } \rho(x, y) > \varepsilon. \end{cases} \tag{2.34}$$

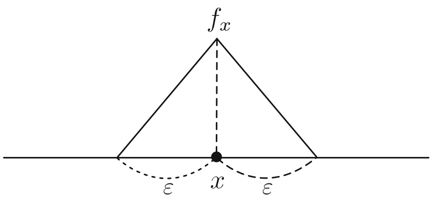

Fig. 2.13.

Let E be an arbitrary compact subset of K. Then, for all sufficiently small $\varepsilon > 0$, the functions f_x, $x \in E$, are in $C_0(K)$ and satisfy the condition

$$\|f_x - f_z\|_\infty \leq \frac{1}{\varepsilon}\rho(x, z) \quad \text{for all } x, z \in E. \tag{2.35}$$

However, for any $\delta > 0$, by the compactness of E we can find a finite number of points $x_1, x_2, \ldots, x_n$ of E such that

$$E = \bigcup_{k=1}^{n} U_{\delta\varepsilon/4}(x_k),$$

and hence

$$\min_{1 \leq k \leq n} \rho(x, x_k) \leq \frac{\delta\varepsilon}{4} \quad \text{for all } x \in E.$$

Thus, by combining this inequality with inequality (2.35) with $z := x_k$ we obtain that

$$\min_{1 \leq k \leq n} \|f_x - f_{x_k}\|_\infty \leq \frac{\delta}{4} \quad \text{for all } x \in E. \tag{2.36}$$

Now we have, by formula (2.34),

$$\begin{aligned} 0 \leq 1 - p_t(x, U_\varepsilon(x)) &\leq 1 - \int_{K_\partial} p_t(x, dy) f_x(y) \\ &= f_x(x) - T_t f_x(x) \\ &\leq \|f_x - T_t f_x\|_\infty \\ &\leq \|f_x - f_{x_k}\|_\infty + \|f_{x_k} - T_t f_{x_k}\|_\infty \\ &\quad + \|T_t f_{x_k} - T_t f_x\|_\infty \\ &\leq 2\|f_x - f_{x_k}\|_\infty + \|f_{x_k} - T_t f_{x_k}\|_\infty \quad \text{for all } x \in E. \end{aligned}$$

In view of inequality (2.36), the first term on the last inequality is bounded by $\delta/2$ for the right choice of k. Furthermore, it follows from the strong continuity (2.30) of $\{T_t\}$ that the second term tends to zero as $t \downarrow 0$ for each $k = 1, 2, \ldots, n$.

Consequently, we have, for all sufficiently small $t > 0$,

$$\sup_{x \in E} \left[1 - p_t(x, U_\varepsilon(x))\right] \leq \delta.$$

This proves the desired condition (2.27), that is, the uniform stochastic continuity of $p_t(x, \cdot)$.

Finally, it remains to verify condition (L). Assume, to the contrary, that:

For some $s > 0$ and some compact $E \subset K$, there exist a positive constant ε_0, a sequence $\{t_k\}$, $t_k \downarrow t$ $(0 \leq t \leq s)$ and a sequence $\{x_k\}$, $x_k \to \partial$, such that

$$p_{t_k}(x_k, E) \geq \varepsilon_0. \tag{2.37}$$

Now we take a relatively compact subset U of K containing E, and let (see Figure 2.14)

$$f(x) = \frac{\rho(x, K \setminus U)}{\rho(x, E) + \rho(x, K \setminus U)}.$$

Then it follows that the function $f(x)$ is in $C_0(K)$ and satisfies the condition

$$T_t f(x) = \int_K p_t(x, dy) f(y) \geq p_t(x, E) \geq 0.$$

Therefore, by combining this inequality with inequality (2.37) we obtain that

$$T_{t_k} f(x_k) \geq p_{t_k}(x_k, E) \geq \varepsilon_0. \tag{2.38}$$

However, we have the inequality

$$T_{t_k} f(x_k) \leq \|T_{t_k} f - T_t f\|_\infty + T_t f(x_k). \tag{2.39}$$

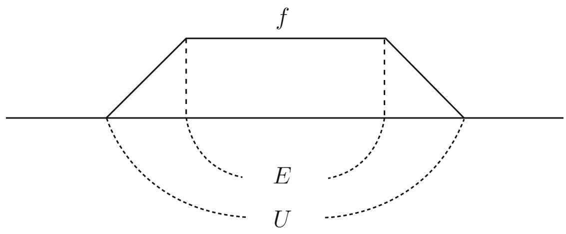

Fig. 2.14.

Since the semigroup $\{T_t\}$ is strongly continuous and since we have the assertion

$$\lim_{k\to\infty} T_t f(x_k) = T_t f(\partial) = 0,$$

we can let $k \to \infty$ in inequality (2.39) to obtain that

$$\limsup_{k\to\infty} T_{t_k} f(x_k) = 0.$$

This contradicts inequality (2.38).

The proof of Theorem 2.14 is now complete. $\square$

A family $\{T_t\}_{t\geq 0}$ of bounded linear operators acting on the space $C_0(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s} = T_t \cdot T_s$, t, $s \geq 0$ (the semigroup property); $T_0 = I$.
(ii) The family $\{T_t\}$ is strongly continuous in t for all $t \geq 0$:

$$\lim_{s\downarrow 0} \|T_{t+s} f - T_t f\|_\infty = 0, \quad f \in C_0(K).$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C_0(K)$:

$$f \in C_0(K),\ 0 \leq f(x) \leq 1 \quad \text{on } K \implies 0 \leq T_t f(x) \leq 1 \quad \text{on } K.$$

Rephrased, Theorem 2.14 gives a characterization of Feller semigroups in terms of Markov transition functions:

Theorem 2.15. *If $p_t(x, \cdot)$ is a uniformly stochastically continuous C_0-transition function on K and satisfies condition (L), then its associated operators $\{T_t\}_{t\geq 0}$, defined by formula (2.28), form a Feller semigroup on K.*

Conversely, if $\{T_t\}_{t\geq 0}$ is a Feller semigroup on K, then there exists a uniformly stochastically continuous C_0-transition $p_t(x, \cdot)$ on K, satisfying condition (L), such that formula (2.28) holds.

The most important applications of Theorem 2.15 are of course in the second statement.

2.2.6 Generation Theorems of Feller Semigroups

In this subsection we prove various generation theorems of Feller semigroups by using the Hille–Yosida theory of semigroups.

If $\{T_t\}_{t \geq 0}$ is a Feller semigroup on K, we define its *infinitesimal generator* A by the formula

$$Au = \lim_{t \downarrow 0} \frac{T_t u - u}{t}, \quad u \in C_0(K), \tag{2.40}$$

provided that the limit (2.40) exists in the space $C_0(K)$. More precisely, the generator A is a linear operator from $C_0(K)$ into itself defined as follows:

(1) The domain $\mathcal{D}(A)$ of A is the set

$$\mathcal{D}(A) = \{u \in C_0(K) : \text{the limit (2.40) exists}\}.$$

(2) $Au = \lim_{t \downarrow 0} \frac{T_t u - u}{t}$, $u \in \mathcal{D}(A)$.

The next theorem is a version of the Hille–Yosida theorem adapted to the present context (cf. [Ta2, Theorem 9.3.1 and Corollary 9.3.2]):

Theorem 2.16 (Hille–Yosida). *(i) Let $\{T_t\}_{t \geq 0}$ be a Feller semigroup on K and let A be its infinitesimal generator. Then we have the following four assertions:*

(a) The domain $\mathcal{D}(A)$ is dense in the space $C_0(K)$.
(b) For each $\alpha > 0$, the equation $(\alpha I - A)u = f$ has a unique solution u in $\mathcal{D}(A)$ for any $f \in C_0(K)$. Hence, for each $\alpha > 0$ the Green operator $(\alpha I - A)^{-1} : C_0(K) \to C_0(K)$ can be defined by the formula

$$u = (\alpha I - A)^{-1} f, \quad f \in C_0(K).$$

(c) For each $\alpha > 0$, the operator $(\alpha I - A)^{-1}$ is non-negative on $C_0(K)$:

$$f \in C_0(K), \ f(x) \geq 0 \quad \text{on } K \implies (\alpha I - A)^{-1} f(x) \geq 0 \quad \text{on } K.$$

(d) For each $\alpha > 0$, the operator $(\alpha I - A)^{-1}$ is bounded on $C_0(K)$ with norm

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha}.$$

(ii) Conversely, if A is a linear operator from $C_0(K)$ into itself satisfying condition (a) and if there is a non-negative constant α_0 such that, for all $\alpha > \alpha_0$, conditions (b) through (d) are satisfied, then A is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on K.

Proof. In view of the Hille–Yosida theory (see [Yo, Chapter IX, Section 7]), it suffices to show that the semigroup $\{T_t\}_{t \geq 0}$ is non-negative if and only if its resolvents (Green operators) $\{(\alpha I - A)^{-1}\}_{\alpha > \alpha_0}$ are non-negative.

The "only if" part is an immediate consequence of the following expression of $(\alpha I - A)^{-1}$ in terms of the semigroup $\{T_t\}$:

$$(\alpha I - A)^{-1} = \int_0^\infty \exp[-\alpha t]\, T_t\, dt, \quad \alpha > 0.$$

On the other hand, the "if" part follows from the expression of the semigroup $T_t(\alpha)$ in terms of the Yosida approximation $J_\alpha = \alpha(\alpha I - A)^{-1}$:

$$T_t(\alpha) = \exp[-\alpha t]\, \exp\left[\alpha t J_\alpha\right] = \exp[-\alpha t] \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!}\, J_\alpha^n,$$

and the definition of the semigroup T_t:

$$T_t = \lim_{\alpha \to \infty} T_t(\alpha).$$

The proof of Theorem 2.16 is complete. $\square$

Corollary 2.17. *Let K be a* compact *metric space and let A be the infinitesimal generator of a Feller semigroup on K. Assume that the constant function* 1 *belongs to the domain $\mathcal{D}(A)$ of A and that we have, for some constant c,*

$$A1(x) \leq -c \quad \text{on } K. \tag{2.41}$$

Then the operator $A' = A + cI$ is the infinitesimal generator of some Feller semigroup on K.

Proof. It follows from an application of part (i) of Theorem 2.16 that, for all $\alpha > c$ the operators

$$(\alpha I - A')^{-1} = ((\alpha - c)I - A)^{-1}$$

are defined and non-negative on the whole space $C(K)$. However, in view of inequality (2.41) we obtain that

$$\alpha \leq \alpha - (A1 + c) = (\alpha I - A')1 \quad \text{on } K,$$

so that

$$\alpha(\alpha I - A')^{-1}1 \leq (\alpha I - A')^{-1}(\alpha I - A')1 = 1 \quad \text{on } K.$$

Hence we have, for all $\alpha > c$,

$$\|(\alpha I - A')^{-1}\| = \|(\alpha I - A')^{-1}1\|_\infty \leq \frac{1}{\alpha}.$$

Therefore, by applying part (ii) of Theorem 2.16 to the operator A' we find that A' is the infinitesimal generator of some Feller semigroup on K.

The proof of Corollary 2.17 is complete. $\square$

Now we write down explicitly the infinitesimal generators of Feller semigroups associated with the transition functions in Examples 2.1 through 2.7 (cf. [DY]).

Example 2.8 (Uniform motion). $K = \mathbf{R}$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K)\}, \\ Af = vf', \quad f \in \mathcal{D}(A). \end{cases}$$

Example 2.9 (Poisson process). $K = \mathbf{R}$ and

$$\begin{cases} \mathcal{D}(A) = C_0(K), \\ Af(x) = \lambda(f(x+1) - f(x)), \quad f \in \mathcal{D}(A). \end{cases}$$

The operator A is not "local"; the value $Af(x)$ depends on the values $f(x)$ and $f(x+1)$. This reflects the fact that the Poisson process changes state by jumps.

Example 2.10 (Brownian motion). $K = \mathbf{R}$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K)\}, \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

The operator A is "local", that is, the value $Af(x)$ is determined by the values of f in an arbitrary small neighborhood of x. This reflects the fact that Brownian motion changes state by continuous motion.

Example 2.11 (Brownian motion with constant drift). $K = \mathbf{R}$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K)\}, \\ Af = \frac{1}{2} f'' + mf', \quad f \in \mathcal{D}(A). \end{cases}$$

Example 2.12 (Cauchy process). $K = \mathbf{R}$ and, the domain $\mathcal{D}(A)$ contains C^2 functions on K with compact support, and the infinitesimal generator A is of the form

$$Af(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} (f(x+y) - f(x)) \frac{dy}{y^2}.$$

The operator A is not "local", which reflects the fact that the Cauchy process changes state by jumps.

Example 2.13 (Reflecting barrier Brownian motion). $K = [0, \infty)$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K),\ f'(0) = 0\}, \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

Example 2.14 (Sticking barrier Brownian motion). $K = [0, \infty)$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K),\ f''(0) = 0\}, \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

Finally, here are two more examples where it is difficult to begin with a transition function and the infinitesimal generator is the basic tool of describing the process.

Example 2.15 (Sticky barrier Brownian motion). $K = [0, \infty)$ and

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K),\ f'(0) - \alpha f''(0) = 0\}, \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

Here α is a positive constant.

This process may be thought of as a "combination" of the reflecting and sticking Brownian motions. The reflecting and sticking cases are obtained by letting $\alpha \to 0$ and $\alpha \to \infty$, respectively.

Example 2.16 (Absorbing barrier Brownian motion). $K = [0, \infty)$ where the boundary point 0 is identified with the point at infinity ∂.

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) : f' \in C_0(K),\ f'' \in C_0(K), f(0) = 0\}, \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

This represents Brownian motion with an absorbing barrier at $x = 0$; a Brownian particle "dies" at the first moment when it hits the boundary $x = 0$. Namely, the point 0 is the terminal point.

It is worth pointing out here that a strong Markov process cannot stay at a single position for a positive length of time and then leave that position by continuous motion; it must either jump away or leave instantaneously.

We give a simple example of a strong Markov process which changes state not by continuous motion but by jumps when the motion reaches the boundary.

Example 2.17. $K = [0, \infty)$.

$$\begin{cases} \mathcal{D}(A) = \{f \in C_0(K) \cap C^2(K) : f' \in C_0(K), f'' \in C_0(K), \\ \quad f''(0) = 2c \int_0^\infty (f(y) - f(0)) dF(y), \\ Af = \frac{1}{2} f'', \quad f \in \mathcal{D}(A). \end{cases}$$

Here c is a positive constant and F is a distribution function on the interval $(0, \infty)$.

This process may be interpreted as follows. When a Brownian particle reaches the boundary $x = 0$, it stays there for a positive length of time and then jumps back to a random point, chosen with the function F, in the interior

$(0, \infty)$. The constant c is the parameter in the "waiting time" distribution at the boundary $x = 0$. We remark that the boundary condition

$$f''(0) = 2c \int_0^\infty (f(y) - f(0))\, dF(y)$$

depends on the values of f far away from the boundary $x = 0$, unlike the boundary conditions in Examples 2.13 through 2.16.

Although Theorem 2.16 asserts precisely when a linear operator A is the infinitesimal generator of some Feller semigroup, it is usually difficult to verify conditions (b) through (d). So we give useful criteria in terms of the *maximum principle* (see [BCP], [SU], [Ra], [Ta2, Theorem 9.3.3 and Corollary 9.3.4]):

Theorem 2.18 (Hille–Yosida–Ray). *Let K be a* compact *metric space. Then we have the following two assertions:*

(i) Let B be a linear operator from $C(K) = C_0(K)$ into itself, and assume that:
  - *(α) The domain $\mathcal{D}(B)$ of B is dense in the space $C(K)$.*
  - *(β) There exists an open and dense subset K_0 of K such that if a function $u \in \mathcal{D}(B)$ takes a* positive *maximum at a point x_0 of K_0, then we have the inequality*

$$Bu(x_0) \leq 0.$$

  Then the operator B is closable in the space $C(K)$.

(ii) Let B be as in part (i), and further assume that:
  - *(β') If a function $u \in \mathcal{D}(B)$ takes a* positive *maximum at a point x' of K, then we have the inequality*

$$Bu(x') \leq 0.$$

  - *(γ) For some $\alpha_0 \geq 0$, the range $\mathcal{R}(\alpha_0 I - B)$ of $\alpha_0 I - B$ is dense in the space $C(K)$.*

Then the minimal closed extension $\overline{B}$ of B is the infinitesimal generator of some Feller semigroup on K.

Proof. (i) It suffices to show that:

$$\{u_n\} \subset \mathcal{D}(B),\ u_n \to 0 \text{ and } Bu_n \to v \quad \text{in } C(K) \implies v = 0.$$

By replacing v by $-v$ if necessary, we assume, to the contrary, that:

The function $v(x)$ takes a positive value at some point of K.

Then, since K_0 is open and dense in K, we can find a point x_0 of K_0, a neighborhood U of x_0 contained in K_0 and a positive constant ε such that we have, for all sufficiently large n,

$$Bu_n(x) > \varepsilon \quad \text{for all } x \in U. \tag{2.42}$$

On the other hand, by condition (α) there exists a function $h \in \mathcal{D}(B)$ such that

$$\begin{cases} h(x_0) > 1, \\ h(x) < 0 \quad \text{for all } x \in K \setminus U. \end{cases}$$

Therefore, since $u_n \to 0$ in $C(K)$, it follows that the function

$$u_n'(x) = u_n(x) + \frac{\varepsilon h(x)}{1 + \|Bh\|_\infty}$$

satisfies the conditions

$$u_n'(x_0) = u_n(x_0) + \frac{\varepsilon h(x_0)}{1 + \|Bh\|_\infty} > 0,$$

$$u_n'(x) = u_n(x) + \frac{\varepsilon h(x)}{1 + \|Bh\|_\infty} < 0 \quad \text{for all } x \in K \setminus U,$$

if n is sufficiently large. This implies that the function $u_n' \in \mathcal{D}(B)$ takes its positive maximum at a point x_n' of $U \subset K_0$. Hence we have, by condition (β),

$$Bu_n'(x_n') \leq 0.$$

However, it follows from inequality (2.42) that

$$Bu_n'(x_n') = Bu_n(x_n') + \varepsilon \frac{Bh(x_n')}{1 + \|Bh\|_\infty} > Bu_n(x_n') - \varepsilon > 0.$$

This is a contradiction.

(ii) We apply part (ii) of Theorem 2.16 to the minimal closed extension $\overline{B}$ of B. The proof is divided into six steps.

Step 1: First, we show that

$$u \in \mathcal{D}(B),\ (\alpha_0 I - B)u \geq 0 \quad \text{on } K \implies u \geq 0 \quad \text{on } K. \tag{2.43}$$

By condition (γ), we can find a function $v \in \mathcal{D}(B)$ such that

$$(\alpha_0 I - B)v \geq 1 \quad \text{on } K. \tag{2.44}$$

Then we have, for any $\varepsilon > 0$,

$$\begin{cases} u + \varepsilon v \in \mathcal{D}(B), \\ (\alpha_0 I - B)(u + \varepsilon v) \geq \varepsilon \quad \text{on } K. \end{cases}$$

In view of condition (β'), this implies that the function $-(u(x) + \varepsilon v(x))$ does not take any positive maximum on K, so that

$$u(x) + \varepsilon v(x) \geq 0 \quad \text{on } K.$$

Thus, by letting $\varepsilon \downarrow 0$ in this inequality we obtain that

$$u(x) \geq 0 \quad \text{on } K.$$

This proves the desired assertion (2.43).

Step 2: It follows from assertion (2.43) that the inverse $(\alpha_0 I - B)^{-1}$ of $\alpha_0 I - B$ is defined and non-negative on the range $\mathcal{R}(\alpha_0 I - B)$. Moreover, it is bounded with norm

$$\|(\alpha_0 I - B)^{-1}\| \leq \|v\|_\infty. \tag{2.45}$$

Here $v(x)$ is the function which satisfies condition (2.44).

Indeed, since $g = (\alpha_0 I - B)v \geq 1$ on K, it follows that, for all $f \in C(K)$,

$$-\|f\|_\infty g \leq f \leq \|f\|_\infty g \quad \text{on } K.$$

Hence, by the non-negativity of $(\alpha_0 I - B)^{-1}$ we have, for all $f \in \mathcal{R}(\alpha_0 I - B)$,

$$-\|f\|_\infty v \leq (\alpha_0 I - B)^{-1} f \leq \|f\|_\infty v \quad \text{on } K.$$

This proves the desired inequality (2.45).

Step 3: Next we show that

$$\mathcal{R}(\alpha_0 I - \overline{B}) = C(K). \tag{2.46}$$

Let $f(x)$ be an arbitrary element of $C(K)$. By condition (γ), we can find a sequence $\{u_n\}$ in $\mathcal{D}(B)$ such that $f_n = (\alpha_0 I - B)u_n \to f$ in $C(K)$. Since the inverse $(\alpha_0 I - B)^{-1}$ is bounded, it follows that $u_n = (\alpha_0 I - B)^{-1} f_n$ converges to some function $u \in C(K)$, and hence $B u_n = \alpha_0 u_n - f_n$ converges to $\alpha_0 u - f$ in $C(K)$. Thus we have, by the closedness of $\overline{B}$,

$$\begin{cases} u \in \mathcal{D}(\overline{B}), \\ \overline{B} u = \alpha_0 u - f, \end{cases}$$

so that

$$(\alpha_0 I - \overline{B}) u = f.$$

This proves the desired assertion (2.46).

Step 4: Furthermore, we show that

$$u \in \mathcal{D}(\overline{B}),\ (\alpha_0 I - \overline{B}) u \geq 0 \quad \text{on } K \implies u \geq 0 \quad \text{on } K. \tag{2.47}$$

Since $\mathcal{R}(\alpha_0 I - \overline{B}) = C(K)$, in view of the proof of assertion (2.47) it suffices to show the following:

If a function $u \in \mathcal{D}(\overline{B})$ takes a positive maximum at a point x' of K, then we have the inequality

$$\overline{B} u(x') \leq 0. \tag{2.48}$$

Assume, to the contrary, that

$$\overline{B} u(x') > 0.$$

Since there exists a sequence $\{u_n\}$ in $\mathcal{D}(B)$ such that $u_n \to u$ and $Bu_n \to \overline{B}u$ in $C(K)$, we can find a neighborhood U of x' and a positive constant ε such that, for all sufficiently large n,

$$Bu_n(x) > \varepsilon \quad \text{for all } x \in U. \tag{2.49}$$

Furthermore, by condition (α) we can find a function $h \in \mathcal{D}(B)$ such that

$$\begin{cases} h(x') > 1, \\ h(x) < 0 \quad \text{for all } x \in K \setminus U. \end{cases}$$

Then it follows that the function

$$u_n'(x) = u_n(x) + \frac{\varepsilon h(x)}{1 + \|Bh\|_\infty}$$

satisfies the condition

$$\begin{cases} u_n'(x') > u(x') > 0, \\ u_n'(x) < u(x') \quad \text{for all } x \in K \setminus U, \end{cases}$$

if n is sufficiently large. This implies that the function $u_n' \in \mathcal{D}(B)$ takes its positive maximum at a point x_n' of U. Hence we have, by condition (β'),

$$Bu_n'(x_n') \leq 0, \quad x_n' \in U.$$

However, it follows from inequality (2.49) that

$$Bu_n'(x_n') = Bu_n(x_n') + \varepsilon \frac{Bh(x_n')}{1 + \|Bh\|_\infty} > Bu_n(x_n') - \varepsilon > 0.$$

This is a contradiction.

Step 5: In view of Steps 3 and 4, we obtain that the inverse $(\alpha_0 I - \overline{B})^{-1}$ of $\alpha_0 I - \overline{B}$ is defined on the whole space $C(K)$, and is bounded with norm

$$\|(\alpha_0 I - \overline{B})^{-1}\| = \|(\alpha_0 I - \overline{B})^{-1} 1\|_\infty.$$

Step 6: Finally, we show that:

For *all* $\alpha > \alpha_0$, the inverse $(\alpha I - \overline{B})^{-1}$ of $\alpha I - \overline{B}$ is defined on the whole space $C(K)$, and is non-negative and bounded with norm

$$\|(\alpha I - \overline{B})^{-1}\| \leq \frac{1}{\alpha}. \tag{2.50}$$

We let

$$G_{\alpha_0} = (\alpha_0 I - \overline{B})^{-1}.$$

First, we choose a constant $\alpha_1 > \alpha_0$ such that

$$(\alpha_1 - \alpha_0)\|G_{\alpha_0}\| < 1,$$

and let

$$\alpha_0 < \alpha \leq \alpha_1.$$

Then, for any $f \in C(K)$, the Neumann series

$$u = \left(I + \sum_{n=1}^{\infty} (\alpha_0 - \alpha)^n G_{\alpha_0}^n \right) G_{\alpha_0} f$$

converges in $C(K)$, and is a solution of the equation

$$u - (\alpha_0 - \alpha) G_{\alpha_0} u = G_{\alpha_0} f.$$

Hence we have the assertions

$$\begin{cases} u \in \mathcal{D}(\overline{B}), \\ (\alpha I - \overline{B})u = f. \end{cases}$$

This proves that

$$\mathcal{R}(\alpha I - \overline{B}) = C(K), \quad \alpha_0 < \alpha \leq \alpha_1. \tag{2.51}$$

Thus, by arguing just as in the proof of Step 1 we obtain that, for any $\alpha_0 < \alpha \leq \alpha_1$,

$$u \in \mathcal{D}(\overline{B}), \ (\alpha I - \overline{B})u \geq 0 \text{ on } K \implies u \geq 0 \text{ on } K. \tag{2.52}$$

By combining assertions (2.51) and (2.52), we find that, for any $\alpha_0 < \alpha \leq \alpha_1$, the inverse $(\alpha I - \overline{B})^{-1}$ is defined and non-negative on the whole space $C(K)$.

We let

$$G_\alpha = (\alpha I - \overline{B})^{-1}, \quad \alpha_0 < \alpha \leq \alpha_1.$$

Then it follows that the operator G_α is bounded with norm

$$\|G_\alpha\| \leq \frac{1}{\alpha}. \tag{2.53}$$

Indeed, in view of assertion (2.48) it follows that if a function $u \in \mathcal{D}(\overline{B})$ takes a positive maximum at a point x' of K, then we have the inequality

$$\overline{B}u(x') \leq 0,$$

so that

$$\max_{x \in K} u(x) = u(x') \leq \frac{1}{\alpha}(\alpha I - \overline{B})u(x') \leq \frac{1}{\alpha}\|(\alpha I - \overline{B})u\|_\infty. \tag{2.54}$$

Similarly, if the function $u \in \mathcal{D}(\overline{B})$ takes a negative minimum at a point of K, then (replacing $u(x)$ by $-u(x)$), we have the inequality

$$-\min_{x \in K} u(x) = \max_{x \in K}(-u(x)) \leq \frac{1}{\alpha}\|(\alpha I - \overline{B})u\|_\infty. \tag{2.55}$$

The desired inequality (2.53) follows from inequalities (2.54) and (2.55).

Summing up, we have proved assertion (2.50) for all $\alpha_0 < \alpha \leq \alpha_1$.

Now we assume that assertion (2.50) is proved for all $\alpha_0 < \alpha \leq \alpha_{n-1}$, $n = 2, 3, \ldots$. Then, by taking

$$\alpha_n = 2\alpha_{n-1} - \frac{\alpha_1}{2}, \quad n \geq 2,$$

or equivalently

$$\alpha_n = \left(2^{n-2} + \frac{1}{2}\right)\alpha_1, \quad n \geq 2,$$

we have, for all $\alpha_{n-1} < \alpha \leq \alpha_n$,

$$\begin{aligned}(\alpha - \alpha_{n-1})\|G_{\alpha_{n-1}}\| &\leq \frac{\alpha - \alpha_{n-1}}{\alpha_{n-1}} \\ &\leq \frac{\alpha_n - \alpha_{n-1}}{\alpha_{n-1}} \\ &= \frac{1}{1 + 2^{2-n}} \\ &< 1.\end{aligned}$$

Hence assertion (2.50) for $\alpha_{n-1} < \alpha \leq \alpha_n$ is proved just as in the proof of assertion (2.50) for $\alpha_0 < \alpha \leq \alpha_1$. This proves the desired assertion (2.50).

Consequently, by applying part (ii) of Theorem 2.16 to the operator $\overline{B}$ we obtain that $\overline{B}$ is the infinitesimal generator of some Feller semigroup on K.

The proof of Theorem 2.18 is now complete. □

Corollary 2.19. *Let A be the infinitesimal generator of a Feller semigroup on a* compact *metric space K and let M be a bounded linear operator on $C(K)$ into itself. Assume that either M or $A' = A + M$ satisfies condition (β'). Then the operator A' is the infinitesimal generator of some Feller semigroup on K.*

Proof. We apply part (ii) of Theorem 2.18 to the operator A'.

First, we remark that $A' = A + M$ is a densely defined, closed linear operator from $C(K)$ into itself. Since the semigroup $\{T_t\}_{t \geq 0}$ is non-negative and contractive on $C(K)$, it follows that if a function $u \in \mathcal{D}(A)$ takes a positive maximum at a point x' of K, then we have the inequality

$$Au(x') = \lim_{t \downarrow 0} \frac{T_t u(x') - u(x')}{t} \leq 0.$$

This implies that if M satisfies condition (β'), so does $A' = A + M$.

We let

$$G_{\alpha_0} = (\alpha_0 I - A)^{-1}, \quad \alpha_0 > 0.$$

If α_0 is so large that

$$\|G_{\alpha_0} M\| \leq \|G_{\alpha_0}\| \cdot \|M\| \leq \frac{\|M\|}{\alpha_0} < 1,$$

then the Neumann series

$$u = \left(I + \sum_{n=1}^{\infty} (G_{\alpha_0} M)^n \right) G_{\alpha_0} f$$

converges in $C(K)$ for any $f \in C(K)$, and is a solution of the equation

$$u - G_{\alpha_0} M u = G_{\alpha_0} f.$$

Hence we have the assertions

$$\begin{cases} u \in \mathcal{D}(A) = \mathcal{D}(A'), \\ (\alpha_0 I - A')u = f. \end{cases}$$

This proves that

$$\mathcal{R}(\alpha_0 I - A') = C(K).$$

Therefore, by applying part (ii) of Theorem 2.18 to the operator A' we obtain that A' is the infinitesimal generator of some Feller semigroup on K.

The proof of Corollary 2.19 is complete. □

3

L^p Theory of Pseudo-Differential Operators

In this chapter we present a brief description of the basic concepts and results of the L^p theory of pseudo-differential operators which may be considered as a modern theory of the classical potential theory. In particular, we formulate the Besov space boundedness theorem due to Bourdaud [Bo] (Theorem 3.15) and a useful criterion for hypoellipticity due to Hörmander [Ho2] (Theorem 3.16) which play an essential role in the proof of our main results. For detailed studies of pseudo-differential operators, the reader is referred to Chazarain–Piriou [CP], Hörmander [Ho3], Kumano-go [Ku] and Taylor [Ty].

3.1 Function Spaces

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an n-dimensional, compact smooth manifold with boundary. We may assume the following (see Figures 3.1 and 3.2):

(a) The domain Ω is a relatively compact open subset of an n-dimensional, compact smooth manifold M without boundary in which Ω has a smooth boundary Γ.
(b) In a neighborhood W of Γ in M a normal coordinate t is chosen so that the points of W are represented as (x', t), $x' \in \Gamma$, $-1 < t < 1$; $t > 0$ in Ω, $t < 0$ in $M \setminus \overline{\Omega}$ and $t = 0$ only on Γ.
(c) The manifold M is equipped with a strictly positive density μ which, on W, is the product of a strictly positive density ω on Γ and the Lebesgue measure dt on $(-1, 1)$. This manifold M is called the *double* of Ω.

The function spaces we shall treat are the following (cf. [AF], [BL], [Ca], [Fr1], [Tb], [Tr]):

(i) The generalized Sobolev spaces $H^{s,p}(\Omega)$ and $H^{s,p}(M)$, consisting of all potentials of order s of L^p functions. When s is integral, these spaces coincide with the usual Sobolev spaces $W^{s,p}(\Omega)$ and $W^{s,p}(M)$, respectively.

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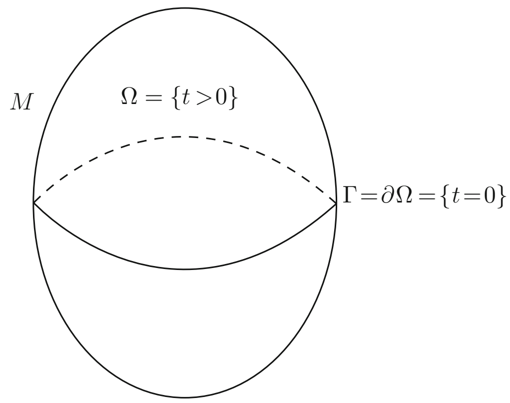

Fig. 3.1.

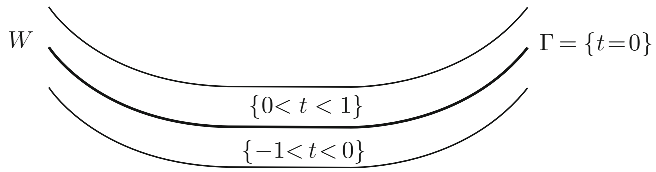

Fig. 3.2.

(ii) The Besov spaces $B^{s,p}(\Gamma)$. These are functions spaces defined in terms of the L^p modulus of continuity, and enter naturally in connection with boundary value problems.

First, if $1 \leq p < \infty$, we let

$L^p(\Omega) =$ the space of (equivalence classes of) Lebesgue measurable functions $u(x)$ on Ω such that $|u(x)|^p$ is integrable on Ω.

The space $L^p(\Omega)$ is a Banach space with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, we let

$L^\infty(\Omega) =$ the space of (equivalence classes of) essentially bounded, Lebesgue measurable functions $u(x)$ on Ω.

The space $L^\infty(\Omega)$ is a Banach space with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|.$$

We recall the basic definitions and facts about the Fourier transform. If $f \in L^1(\mathbf{R}^n)$, we define its (direct) Fourier transform $\mathcal{F}f(\xi)$ by the formula

$$\mathcal{F}f(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n),$$

where $i = \sqrt{-1}$ and $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n$. We also denote $\mathcal{F}f(\xi)$ by $\widehat{f}(\xi)$.

Similarly, if $g \in L^1(\mathbf{R}^n)$, we define its inverse Fourier transform $\mathcal{F}^*g(x)$ by the formula

$$\mathcal{F}^*g(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} g(\xi)\, d\xi.$$

We also denote $\mathcal{F}^*g(x)$ by $\check{g}(x)$.

We introduce a subspace of $L^1(\mathbf{R}^n)$ which is invariant under the Fourier transform. We define the Schwartz space

$\mathcal{S}(\mathbf{R}^n) =$ the space of smooth functions $\varphi(x)$ rapidly decreasing at infinity on $\mathbf{R}^n$ such that we have, for any non-negative integer j,

$$p_j(\varphi) = \sup_{\substack{x \in \mathbf{R}^n \\ |\alpha| \le j}} \left\{ (1 + |x|^2)^{j/2} |\partial^\alpha \varphi(x)| \right\} < \infty.$$

We equip the space $\mathcal{S}(\mathbf{R}^n)$ with the topology defined by the countable family $\{p_j\}$ of seminorms. The space $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space. The transforms $\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}(\mathbf{R}^n)$ continuously into itself, and $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I$ on $\mathcal{S}(\mathbf{R}^n)$.

Example 3.1. For $a > 0$, we have the assertion

$$\varphi(x) = e^{-a|x|^2} \in \mathcal{S}(\mathbf{R}^n).$$

Furthermore, it is easy to verify the following formulas:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} e^{-a|x|^2}\, dx = \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{|x|^2}{4a}},$$

$$\mathcal{F}^*(\mathcal{F}\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widehat{\varphi}(\xi)\, d\xi = \varphi(x).$$

Since the injection of $C_0^\infty(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^n)$ is continuous, it follows that the dual space $\mathcal{S}'(\mathbf{R}^n)$ of $\mathcal{S}(\mathbf{R}^n)$ consists of those distributions $T \in \mathcal{D}'(\mathbf{R}^n)$ that have continuous extensions to $\mathcal{S}(\mathbf{R}^n)$. The elements of $\mathcal{S}'(\mathbf{R}^n)$ are called *tempered distributions* on $\mathbf{R}^n$.

Roughly speaking, the tempered distributions are those which grow at most polynomially at infinity, since the functions in $\mathcal{S}(\mathbf{R}^n)$ die out faster than any power of x at infinity. More precisely, we have the following examples of tempered distributions:

(1) The functions in $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, are tempered distributions.

(2) A locally integrable function on $\mathbf{R}^n$ is a tempered distribution if it grows at most polynomially at infinity.
(3) If $u \in \mathcal{S}'(\mathbf{R}^n)$ and $f(x)$ is a smooth function on $\mathbf{R}^n$ all of whose derivatives grow at most polynomially at infinity, then the product $f(x)u(x)$ is a tempered distribution.
(4) Any derivative of a tempered distribution is also a tempered distribution.

Now we give some concrete and important examples of distributions in the space $\mathcal{S}'(\mathbf{R}^n)$:

Example 3.2. (a) The Dirac measure: $\delta(x)$.
(b) Riesz potentials:

$$R_\alpha(x) = \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} \frac{1}{|x|^{n-\alpha}}, \quad 0 < \alpha < n.$$

(c) Newtonian potentials:

$$N(x) = \frac{\Gamma((n-2)/2)}{4\pi^{n/2}} \frac{1}{|x|^{n-2}}, \quad n \geq 3.$$

(d) Bessel potentials:

$$G_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \frac{1}{(4\pi)^{n/2}} \int_0^\infty e^{-t-\frac{|x|^2}{4t}} t^{\frac{\alpha-n}{2}} \frac{dt}{t}, \quad \alpha > 0.$$

It is known (see [AS]) that the function $G_\alpha(x)$ is represented as follows:

$$G_\alpha(x) = \frac{1}{2^{(n+\alpha-2)/2} \pi^{n/2} \Gamma(\alpha/2)} K_{(n-\alpha)/2}(|x|) \, |x|^{\frac{\alpha-n}{2}},$$

where $K_{(n-\alpha)/2}(z)$ is the modified Bessel function of the third kind (cf. [Wt]).
(e) Riesz kernels:

$$R_j(x) = \sqrt{-1} \, \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \, \text{v.p.} \, \frac{x_j}{|x|^{n+1}}, \quad 1 \leq j \leq n.$$

The distribution $\text{v.p.}\,(x_j/|x|^{n+1})$ is an extension of $\text{v.p.}\,(1/x)$ to the n-dimensional case.

The importance of tempered distributions lies in the fact that they have Fourier transforms.

The direct and inverse Fourier transforms can be extended to the space $\mathcal{S}'(\mathbf{R}^n)$ by the following formulas:

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle, \qquad u \in \mathcal{S}'(\mathbf{R}^n), \quad \varphi \in \mathcal{S}(\mathbf{R}^n).$$
$$\langle \mathcal{F}^* v, \varphi \rangle = \langle v, \mathcal{F}^* \varphi \rangle, \quad v \in \mathcal{S}'(\mathbf{R}^n), \quad \varphi \in \mathcal{S}(\mathbf{R}^n).$$

Once again, the transforms $\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}'(\mathbf{R}^n)$ continuously into itself, and $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I$ on $\mathcal{S}'(\mathbf{R}^n)$.

We can calculate explicitly the Fourier transform of the tempered distributions in Example 3.2 as follows:

Example 3.3. (a) The Dirac measure: $(\mathcal{F}\delta)(\xi) = 1$.
(b) Riesz potentials:

$$(\mathcal{F}R_\alpha)(\xi) = \frac{1}{|\xi|^\alpha}, \quad 0 < \alpha < n.$$

(c) Newtonian potentials:

$$(\mathcal{F}N)(\xi) = \frac{1}{|\xi|^2}, \quad n \geq 3.$$

(d) Bessel potentials:

$$(\mathcal{F}G_\alpha)(\xi) = \frac{1}{(1 + |\xi|^2)^{\alpha/2}}, \quad \alpha > 0.$$

(e) Riesz kernels:

$$(\mathcal{F}R_j)(\xi) = \frac{\xi_j}{|\xi|}, \quad 1 \leq j \leq n.$$

If $s \in \mathbf{R}$, we define a linear map

$$J^s : \mathcal{S}'(\mathbf{R}^n) \longrightarrow \mathcal{S}'(\mathbf{R}^n)$$

by the formula

$$J^s u = \mathcal{F}^* \left((1 + |\xi|^2)^{-s/2} \mathcal{F}u \right), \quad u \in \mathcal{S}'(\mathbf{R}^n).$$

This can be visualized as follows:

$$\begin{array}{ccc} u \in \mathcal{S}'(\mathbf{R}^n) & \xrightarrow{J^s} & \mathcal{S}'(\mathbf{R}^n) \ni J^s u \\ {\scriptstyle \mathcal{F}}\downarrow & & \uparrow{\scriptstyle \mathcal{F}^*} \\ \mathcal{F}u \in \mathcal{S}'(\mathbf{R}^n) & \xrightarrow[(1+|\xi|^2)^{-s/2}]{} & \mathcal{S}'(\mathbf{R}^n) \ni (1 + |\xi|^2)^{-s/2} \mathcal{F}u \end{array}$$

Then it is easy to see that the map J^s is an isomorphism of $\mathcal{S}'(\mathbf{R}^n)$ onto itself and that its inverse is the map J^{-s}. The function $J^s u$ is called the *Bessel potential* of order s of u.

(I) Now, if $s \in \mathbf{R}$ and $1 < p < \infty$, we let

$$H^{s,p}(\mathbf{R}^n) = \text{the image of } L^p(\mathbf{R}^n) \text{ under the mapping } J^s.$$

We equip $H^{s,p}(\mathbf{R}^n)$ with the norm $\|u\|_{s,p} = \|J^{-s}u\|_p$ for $u \in H^{s,p}(\mathbf{R}^n)$. The space $H^{s,p}(\mathbf{R}^n)$ is called the (generalized) *Sobolev space* of order s.

We list some basic topological properties of $H^{s,p}(\mathbf{R}^n)$:

(1) The Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is dense in each $H^{s,p}(\mathbf{R}^n)$.

(2) The space $H^{-s,p'}(\mathbf{R}^n)$ is the dual space of $H^{s,p}(\mathbf{R}^n)$, where $p' = p/(p-1)$ is the exponent conjugate to p.

(3) If $s > t$, then we have the inclusions

$$\mathcal{S}(\mathbf{R}^n) \subset H^{s,p}(\mathbf{R}^n) \subset H^{t,p}(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n),$$

with continuous injections.

(4) If s is a non-negative integer, then the space $H^{s,p}(\mathbf{R}^n)$ is isomorphic to the usual Sobolev space $W^{s,p}(\mathbf{R}^n)$, that is, the space $H^{s,p}(\mathbf{R}^n)$ coincides with the space of functions $u \in L^p(\mathbf{R}^n)$ such that $D^\alpha u \in L^p(\mathbf{R}^n)$ for $|\alpha| \leq s$, and the norm $\|\cdot\|_{s,p}$ is equivalent to the norm

$$\left(\sum_{|\alpha| \leq s} \int_{\mathbf{R}^n} |D^\alpha u(x)|^p dx \right)^{1/p}.$$

(II) Next, if $1 < p < \infty$, we let

$$B^{1,p}(\mathbf{R}^{n-1}) = \text{ the space of (equivalence classes of) functions } \varphi(x') \in L^p(\mathbf{R}^{n-1}) \text{ for which the integral } \iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|\varphi(x'+y') - 2\varphi(x') + \varphi(x'-y')|^p}{|y'|^{n-1+p}} \, dy' \, dx' \text{ is finite.}$$

The space $B^{1,p}(\mathbf{R}^{n-1})$ is a Banach space with respect to the norm

$$|\varphi|_{1,p} = \left(\int_{\mathbf{R}^{n-1}} |\varphi(x')|^p \, dx' + \iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|\varphi(x'+y') - 2\varphi(x') + \varphi(x'-y')|^p}{|y'|^{n-1+p}} \, dy' \, dx' \right)^{1/p}.$$

If $p = \infty$, we let

$$B^{1,\infty}(\mathbf{R}^{n-1}) = \text{ the space of (equivalence classes of) functions } \varphi(x') \in L^\infty(\mathbf{R}^{n-1}) \text{ for which the quantity } \sup_{|y'|>0} \frac{\|\varphi(\cdot+y') - 2\varphi(\cdot) + \varphi(\cdot-y')\|_\infty}{|y'|} \text{ is finite.}$$

The space $B^{1,\infty}(\mathbf{R}^{n-1})$ is a Banach space with respect to the norm

$$|\varphi|_{1,\infty} = \|\varphi\|_\infty + \sup_{|y'|>0} \frac{\|\varphi(\cdot+y') - 2\varphi(\cdot) + \varphi(\cdot-y')\|_\infty}{|y'|}.$$

If $s \in \mathbf{R}$, we let

$$B^{s,p}(\mathbf{R}^{n-1}) = \text{ the image of } B^{1,p}(\mathbf{R}^{n-1}) \text{ under the mapping } {J'}^{s-1}, \text{ where } {J'}^{s-1} \text{ is the Bessel potential of order } s-1 \text{ on } \mathbf{R}^{n-1}.$$

We equip the space $B^{s,p}(\mathbf{R}^{n-1})$ with the norm $|\varphi|_{s,p} = \left|{J'}^{-s+1}\varphi\right|_{1,p}$ for $\varphi(x') \in B^{s,p}(\mathbf{R}^{n-1})$. The space $B^{s,p}(\mathbf{R}^{n-1})$ is called the *Besov space* of order s.

We list some basic topological properties of $B^{s,p}(\mathbf{R}^{n-1})$:

(1) The Schwartz space $\mathcal{S}(\mathbf{R}^{n-1})$ is dense in each $B^{s,p}(\mathbf{R}^{n-1})$.
(2) The space $B^{-s,p'}(\mathbf{R}^{n-1})$ is the dual space of $B^{s,p}(\mathbf{R}^{n-1})$, where $p' = p/(p-1)$ is the exponent conjugate to p.
(3) If $s > t$, then we have the inclusions

$$\mathcal{S}(\mathbf{R}^{n-1}) \subset B^{s,p}(\mathbf{R}^{n-1}) \subset B^{t,p}(\mathbf{R}^{n-1}) \subset \mathcal{S}'(\mathbf{R}^{n-1}),$$

with continuous injections.
(4) If $s = m + \sigma$ where m is a non-negative integer and $0 < \sigma < 1$, then the Besov space $B^{s,p}(\mathbf{R}^{n-1})$ coincides with the space of functions $\varphi(x') \in H^{m,p}(\mathbf{R}^{n-1})$ such that, for $|\alpha| = m$, the integral (Slobodeckiĭ seminorm)

$$\iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|D^\alpha\varphi(x') - D^\alpha\varphi(y')|^p}{|x'-y'|^{n-1+p\sigma}}\, dx'\, dy' < \infty.$$

Furthermore, the norm $|\varphi|_{s,p}$ is equivalent to the norm

$$\left(\sum_{|\alpha|\le m} \int_{\mathbf{R}^{n-1}} |D^\alpha\varphi(x')|^p\, dx' + \sum_{|\alpha|=m} \iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|D^\alpha\varphi(x') - D^\alpha\varphi(y')|^p}{|x'-y'|^{n-1+p\sigma}}\, dx'\, dy'\right)^{1/p}.$$

Now we define the generalized Sobolev spaces $H^{s,p}(\Omega)$, $H^{s,p}(M)$ and the Besov spaces $B^{s,p}(\Gamma)$ for arbitrary values of s.

For each $s \in \mathbf{R}$, we define

$$H^{s,p}(\Omega) = \text{ the space of distributions } u \in \mathcal{D}'(\Omega) \text{ such that there exists a function } U \in H^{s,p}(\mathbf{R}^n) \text{ with } U|_\Omega = u,$$

and equip the space $H^{s,p}(\Omega)$ with the norm

$$\|u\|_{s,p} = \inf \|U\|_{s,p},$$

where the infimum is taken over all such U. The space $H^{s,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{s,p}$. We remark that

$$H^{0,p}(\Omega) = L^p(\Omega); \quad \|\cdot\|_{0,p} = \|\cdot\|_p.$$

The spaces $H^{s,p}(M)$ are defined to be locally the spaces $H^{s,p}(\mathbf{R}^n)$, upon using local coordinate systems flattening out M, together with a partition of unity. The spaces $B^{s,p}(\Gamma)$ are defined similarly, with $H^{s,p}(\mathbf{R}^n)$ replaced by $B^{s,p}(\mathbf{R}^{n-1})$. The norms of $H^{s,p}(M)$ and $B^{s,p}(\Gamma)$ will be denoted by $\|\cdot\|_{s,p}$ and $|\cdot|_{s,p}$, respectively.

We state two important theorems that will be used in the study of boundary value problems in the framework of Sobolev spaces of L^p type (see [AF], [BL], [St1], [Tr]):

(I) (**The trace theorem**) Let $1 < p < \infty$. Then the trace map

$$\begin{aligned} \rho : \quad H^{s,p}(\Omega) &\longrightarrow B^{s-1/p,p}(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned}$$

is continuous for all $s > 1/p$, and is surjective.

(II) (**The Rellich–Kondrachov theorem**) If $s > t$, then the injections

$$\begin{aligned} H^{s,p}(M) &\longrightarrow H^{t,p}(M), \\ B^{s,p}(\partial\Omega) &\longrightarrow B^{t,p}(\partial\Omega) \end{aligned}$$

are both compact (or completely continuous).

Finally, we introduce a space of distributions on Ω which behave locally just like the distributions in $H^{s,p}(\mathbf{R}^n)$:

$$\begin{aligned} H^{s,p}_{\mathrm{loc}}(\Omega) = {} & \text{the space of distributions } u \in \mathcal{D}'(\Omega) \text{ such that} \\ & \varphi u \in H^{s,p}(\mathbf{R}^n) \text{ for all } \varphi \in C_0^\infty(\Omega). \end{aligned}$$

We equip the localized Sobolev space $H^{s,p}_{\mathrm{loc}}(\Omega)$ with the topology defined by the seminorms $u \mapsto \|\varphi u\|_{s,p}$ as φ ranges over $C_0^\infty(\Omega)$. It is easy to verify that $H^{s,p}_{\mathrm{loc}}(\Omega)$ is a Fréchet space. The localized Besov space $B^{s,p}_{\mathrm{loc}}(\partial\Omega)$ is defined similarly, with $H^{s,p}(\mathbf{R}^n)$ replaced by $B^{s,p}(\mathbf{R}^{n-1})$.

3.2 Fourier Integral Operators

In this section, we present a brief description of basic concepts and results of the theory of Fourier integral operators.

3.2.1 Symbol Classes

Let Ω be an open subset of $\mathbf{R}^n$. If $m \in \mathbf{R}$ and $0 \le \delta < \rho \le 1$, we let

$$\begin{aligned} S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N) = {} & \text{the set of all functions } a(x,\theta) \in C^\infty(\Omega \times \mathbf{R}^N) \\ & \text{with the property that, for any compact } K \subset \Omega \text{ and} \\ & \text{any multi-indices } \alpha,\ \beta, \text{ there exists a positive constant} \\ & C_{K,\alpha,\beta} \text{ such that we have, for all } x \in K \text{ and } \theta \in \mathbf{R}^N, \\ & \quad \left|\partial_\theta^\alpha \partial_x^\beta a(x,\theta)\right| \le C_{K,\alpha,\beta}(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|}. \end{aligned}$$

The elements of $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N)$ are called *symbols* of order m. We drop the $\Omega \times \mathbf{R}^N$ and use $S^m_{\rho,\delta}$ when the context is clear.

Example 3.4. (1) A polynomial $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ of order m with coefficients in $C^\infty(\Omega)$ is in $S^m_{1,0}(\Omega \times \mathbf{R}^n)$.
(2) If $m \in \mathbf{R}$, the function

$$\Omega \times \mathbf{R}^n \ni (x,\xi) \longmapsto \left(1 + |\xi|^2\right)^{m/2}$$

is in $S^m_{1,0}(\Omega \times \mathbf{R}^n)$.
(3) A function $a(x,\theta) \in C^\infty(\Omega \times (\mathbf{R}^N \setminus \{0\})$ is said to be *positively homogeneous* of degree m in θ if it satisfies the condition

$$a(x,t\theta) = t^m\, a(x,\theta) \quad \text{for all } t > 0 \text{ and } \theta \in \mathbf{R}^N \setminus \{0\}.$$

If $a(x,\theta)$ is positively homogeneous of degree m in θ and if $\varphi(\theta)$ is a smooth function such that $\varphi(\theta) = 0$ for $|\theta| \le 1/2$ and $\varphi(\theta) = 1$ for $|\theta| \ge 1$, then the function $\varphi(\theta)a(x,\theta)$ is in $S^m_{1,0}(\Omega \times \mathbf{R}^N)$.

If K is a compact subset of Ω and if j is a non-negative integer, we define a seminorm $p_{K,j,m}$ on $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N)$ by the formula

$$S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N) \ni a \longmapsto p_{K,j,m}(a) = \sup_{\substack{x\in K,\ \theta\in\mathbf{R}^N \\ |\alpha|\le j}} \frac{\left|\partial_\theta^\alpha \partial_x^\beta a(x,\theta)\right|}{(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|}}.$$

We equip the space $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N)$ with the topology defined by the family $\{p_{K,j,m}\}$ of seminorms where K ranges over all compact subsets of Ω and $j = 0, 1, \ldots$. The space $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N)$ is a Fréchet space.

We set

$$S^{-\infty}(\Omega \times \mathbf{R}^N) = \bigcap_{m\in\mathbf{R}} S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N).$$

The next theorem gives a meaning to a formal sum of symbols of decreasing order:

Theorem 3.1. *Let $a_j(x,\theta) \in S^{m_j}_{\rho,\delta}(\Omega \times \mathbf{R}^N)$, $m_j \downarrow -\infty$, $j = 0, 1, \ldots$. Then there exists a symbol $a(x,\theta) \in S^{m_0}_{\rho,\delta}(\Omega \times \mathbf{R}^N)$, unique modulo $S^{-\infty}(\Omega \times \mathbf{R}^N)$, such that we have, for all positive integer k,*

$$a(x,\theta) - \sum_{j=0}^{k-1} a_j(x,\theta) \in S^{m_k}_{\rho,\delta}(\Omega \times \mathbf{R}^N). \tag{3.1}$$

If formula (3.1) holds true, we write

$$a(x,\theta) \sim \sum_{j=0}^{\infty} a_j(x,\theta).$$

The formal sum $\sum_{j=0}^{\infty} a_j(x,\theta)$ is called an asymptotic expansion of $a(x,\theta)$.

A symbol $a(x,\theta) \in S^m_{1,0}(\Omega \times \mathbf{R}^N)$ is said to be *classical* if there exist smooth functions $a_j(x,\theta)$, positively homogeneous of degree $m-j$ in θ for $|\theta| \geq 1$, such that we have, for all positive integer k,

$$a(x,\theta) - \sum_{j=0}^{k-1} a_j(x,\theta) \in S^{m-k}_{1,0}(\Omega \times \mathbf{R}^N).$$

The homogeneous function $a_0(x,\theta)$ of degree m is called the *principal part* of $a(x,\theta)$.

We let

$$S^m_{\mathrm{cl}}(\Omega \times \mathbf{R}^N) = \text{the set of all classical symbols of order } m.$$

For example, the symbols in Example 3.4 are all classical, and they have respectively as principal part the following functions:

(1) $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.
(2) $|\xi|^m$.
(3) $a(x,\theta)$.

A symbol $a(x,\theta)$ in $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^N)$ is said to be *elliptic* of order m if, for any compact $K \subset \Omega$, there exists a positive constant C_K such that

$$|a(x,\theta)| \geq C_K(1+|\theta|)^m \quad \text{for all } x \in K \text{ and } |\theta| \geq \frac{1}{C_K}.$$

There is a simple criterion in the case of classical symbols.

Theorem 3.2. *Let $a(x,\theta)$ be in $S^m_{\mathrm{cl}}(\Omega \times \mathbf{R}^N)$ with principal part $a_0(x,\theta)$. Then $a(x,\theta)$ is elliptic if and only if it satisfies the condition*

$$a_0(x,\theta) \neq 0 \quad \text{for all } x \in \Omega \text{ and } |\theta| = 1.$$

3.2.2 Phase Functions

Let Ω be an open subset of $\mathbf{R}^n$. A function $\varphi(x,\theta)$ in $C^\infty\left(\Omega \times \left(\mathbf{R}^N \setminus \{0\}\right)\right)$ is called a *phase function* on $\Omega \times (\mathbf{R}^N \setminus \{0\})$ if it satisfies the following three conditions:

(a) $\varphi(x,\theta)$ is real-valued.
(b) $\varphi(x,\theta)$ is positively homogeneous of degree one in the variable θ.
(c) The differential $d\varphi(x,\theta)$ does not vanish on $\Omega \times \left(\mathbf{R}^N \setminus \{0\}\right)$.

Example 3.5. Let U be an open subset of $\mathbf{R}^p$ and $\Omega = U \times U$. The function

$$\varphi(x,y,\xi) = (x-y)\cdot\xi$$

is a phase function on the space $\Omega \times (\mathbf{R}^p \setminus \{0\})$ with $n = 2p$ and $N = p$.

The next lemma will play a fundamental role in defining oscillatory integrals.

Lemma 3.3. *If* $\varphi(x,\theta)$ *is a phase function on* $\Omega \times \left(\mathbf{R}^N \setminus \{0\}\right)$*, then there exists a first-order differential operator*

$$L = \sum_{j=1}^{N} a_j(x,\theta) \frac{\partial}{\partial \theta_j} + \sum_{k=1}^{n} b_k(x,\theta) \frac{\partial}{\partial x_k} + c(x,\theta)$$

such that

$$L(e^{i\varphi}) = e^{i\varphi},$$

and that its coefficients $a_j(x,\theta)$*,* $b_k(x,\theta)$*,* $c(x,\theta)$ *enjoy the following properties:*

$$a_j(x,\theta) \in S^0_{1,0};\ b_k(x,\theta),\ c(x,\theta) \in S^{-1}_{1,0}.$$

Furthermore, the transpose L' *of* L *has coefficients* $a'_j(x,\theta)$*,* $b'_k(x,\theta)$*,* $c'(x,\theta)$ *in the same symbol classes as* $a_j(x,\theta)$*,* $b_k(x,\theta)$*,* $c(x,\theta)$*, respectively.*

For example, if $\varphi(x,y,\xi)$ is a phase function as in Example 3.5

$$\varphi(x,y,\xi) = (x-y)\cdot\xi, \quad (x,y) \in U \times U,\ \xi \in (\mathbf{R}^p \setminus \{0\}),$$

then the operator L is given by the formula

$$L = \frac{1}{\sqrt{-1}} \frac{1-\rho(\xi)}{2+|x-y|^2} \left\{ \sum_{j=1}^{p} (x_j - y_j) \frac{\partial}{\partial \xi_j} + \sum_{k=1}^{p} \frac{\xi_k}{|\xi|^2} \frac{\partial}{\partial x_k} + \sum_{k=1}^{p} \frac{-\xi_k}{|\xi|^2} \frac{\partial}{\partial y_k} \right\} + \rho(\xi),$$

where $\rho(\xi)$ is a function in $C_0^\infty(\mathbf{R}^p)$ such that $\rho(\xi) = 1$ for $|\xi| \leq 1$.

3.2.3 Oscillatory Integrals

If Ω is an open subset of $\mathbf{R}^n$, we let

$$S^\infty_{\rho,\delta}\left(\Omega \times \mathbf{R}^N\right) = \bigcup_{m \in \mathbf{R}} S^m_{\rho,\delta}\left(\Omega \times \mathbf{R}^N\right).$$

If $\varphi(x,\theta)$ is a phase function on $\Omega \times \left(\mathbf{R}^N \setminus \{0\}\right)$, we wish to give a meaning to the integral

$$I_\varphi(aw) = \iint_{\Omega \times \mathbf{R}^N} e^{i\varphi(x,\theta)} a(x,\theta) u(x)\, dx\, d\theta, \quad u \in C_0^\infty(\Omega), \tag{3.2}$$

for each symbol $a(x,\theta) \in S^\infty_{\rho,\delta}\left(\Omega \times \mathbf{R}^N\right)$.

By Lemma 3.3, we can replace $e^{i\varphi}$ in formula (3.2) by $L(e^{i\varphi})$. Then a *formal* integration by parts gives us that

$$I_\varphi(au) = \iint_{\Omega\times\mathbf{R}^N} e^{i\varphi(x,\theta)}\, L'(a(x,\theta)w(x,y))\, dx\, d\theta.$$

However, the properties of the coefficients of the transpose L' imply that L' maps $S^r_{\rho,\delta}$ continuously into $S^{r-\eta}_{\rho,\delta}$ for all $r \in \mathbf{R}$, where $\eta = \min(\rho, 1-\delta)$. By continuing this process, we can reduce the growth of the integrand at infinity until it becomes integrable, and give a meaning to the integral (3.2) for each symbol $a(x,\theta) \in S^\infty_{\rho,\delta}(\Omega \times \mathbf{R}^n)$.

More precisely, we have the following:

Theorem 3.4. *(i) The linear functional*

$$S^{-\infty}\left(\Omega \times \mathbf{R}^N\right) \ni a \longmapsto I_\varphi(au) \in \mathbf{C}$$

extends uniquely to a linear functional ℓ on $S^\infty_{\rho,\delta}\left(\Omega\times\mathbf{R}^N\right)$ whose restriction to each $S^m_{\rho,\delta}\left(\Omega\times\mathbf{R}^N\right)$ is continuous. Furthermore, the restriction of the linear functional ℓ to $S^m_{\rho,\delta}\left(\Omega\times\mathbf{R}^N\right)$ is expressed as the formula

$$\ell(a) = \iint_{\Omega\times\mathbf{R}^N} e^{i\varphi(x,\theta)}\, (L')^k(a(x,\theta)w(x,y))\, dx\, d\theta,$$

where $k > (m+N)/\eta$ and $\eta = \min(\rho, 1-\delta)$.

(ii) For any fixed $a(x,\theta) \in S^m_{\rho,\delta}\left(\Omega\times\mathbf{R}^N\right)$, the mapping

$$C_0^\infty(\Omega) \ni u \longmapsto I_\varphi(au) = \ell(a) \in \mathbf{C} \tag{3.3}$$

is a distribution of order $\leq k$ for $k > (m+N)/\eta$ with $\eta = \min(\rho, 1-\delta)$.

We call the linear functional ℓ on $S^\infty_{\rho,\delta}$ an *oscillatory integral*, but use the standard notation as in formula (3.2). The distribution (3.3) is called the *Fourier integral distribution* associated with the phase function $\varphi(x,\theta)$ and the amplitude $a(x,\theta)$, and will be denoted as follows:

$$K(x) = \int_{\mathbf{R}^N} e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta.$$

If u is a distribution on Ω, then the *singular support* of u is the smallest closed subset of Ω outside of which u is smooth. The singular support of u is denoted by $\operatorname{sing\,supp} u$.

The next theorem estimates the singular support of a Fourier integral distribution.

Theorem 3.5. *If $\varphi(x,\theta)$ is a phase function on the space $\Omega \times \left(\mathbf{R}^N \setminus \{0\}\right)$ and if $a(x,\theta)$ is in $S^\infty_{\rho,\delta}\left(\Omega\times\mathbf{R}^N\right)$, then the distribution*

$$K(x) = \int_{\mathbf{R}^N} e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta \in \mathcal{D}'(\Omega)$$

satisfies the condition

$$\operatorname{sing\,supp} K \subset \left\{x \in \Omega : d_\theta\varphi(x\theta) = 0 \text{ for some } \theta \in \mathbf{R}^N \setminus \{0\}\right\}.$$

3.2.4 Fourier Integral Operators

Let U and V be open subsets of $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively. If $\varphi(x, y, \theta)$ is a phase function on $U \times V \times (\mathbf{R}^N \setminus \{0\})$ and if $a(x, y, \theta) \in S^{\infty}_{\rho,\delta}(U \times V \times \mathbf{R}^N)$, then there is associated a distribution $K \in \mathcal{D}'(U \times V)$ defined by the formula

$$K(x, y) = \int_{\mathbf{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta)\, d\theta.$$

By applying Theorem 3.5 to our situation, we obtain that

$$\operatorname{sing\,supp} K \subset \left\{(x, y) \in U \times V : d_\theta \varphi(x, y, \theta) = 0 \text{ for some } \theta \in \mathbf{R}^N \setminus \{0\}\right\}.$$

The distribution $K \in \mathcal{D}'(U \times V)$ defines a continuous linear operator

$$A : C_0^\infty(V) \longrightarrow \mathcal{D}'(U)$$

by the formula

$$\langle Av, u\rangle = \langle K, u \otimes v\rangle\,, \quad u \in C_0^\infty(U),\ v \in C_0^\infty(V).$$

The operator A is called the *Fourier integral operator* associated with the phase function $\varphi(x, y, \theta)$ and the amplitude $a(x, y, \theta)$, and will be denoted as follows:

$$Av(x) = \iint_{V \times \mathbf{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) v(y)\, dy\, d\theta, \quad v \in C_0^\infty(V).$$

The next theorem summarizes some basic properties of the operator A:

Theorem 3.6. *(i) If $d_{y,\theta}\varphi(x, y, \theta) \neq 0$ on $U \times V \times \left(\mathbf{R}^N \setminus \{0\}\right)$, then the operator A maps $C_0^\infty(V)$ continuously into $C^\infty(U)$.*

(ii) If $d_{x,\theta}\varphi(x, y, \theta) \neq 0$ on $U \times V \times \left(\mathbf{R}^N \setminus \{0\}\right)$, then the operator A extends to a continuous linear operator on $\mathcal{E}'(V)$ into $\mathcal{D}'(U)$.

(iii) If $d_{y,\theta}\varphi(x, y, \theta) \neq 0$ and $d_{x,\theta}\varphi(x, y, \theta) \neq 0$ on $U \times V \times \left(\mathbf{R}^N \setminus \{0\}\right)$, then we have, for all $v \in \mathcal{E}'(V)$,

$$\begin{aligned}&\operatorname{sing\,supp} Av \\ &\subset \left\{x \in U : d_\theta\varphi(x, y, \theta) = 0 \textit{ for some } y \in \operatorname{sing\,supp} v \textit{ and } \theta \in \mathbf{R}^N \setminus \{0\}\right\}.\end{aligned}$$

3.3 Pseudo-Differential Operators

Let Ω_1 and Ω_2 be open subsets of $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively. If $K(x_1, x_2)$ is a distribution in $\mathcal{D}'(\Omega_1 \times \Omega_2)$, we can define a continuous linear operator

$$A \in L(C_0^\infty(\Omega_2), \mathcal{D}'(\Omega_1))$$

by the formula

$$\langle A\psi, \varphi\rangle = \langle K, \varphi \otimes \psi\rangle, \quad \varphi \in C_0^\infty(\Omega_1),\ \psi \in C_0^\infty(\Omega_2).$$

We then write $A = \mathrm{Op}\,(K)$. Since the tensor space $C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$ is sequentially dense in $C_0^\infty(\Omega_1 \times \Omega_2)$, it follows that the mapping

$$\mathcal{D}'(\Omega_1 \times \Omega_2) \ni K \longmapsto \mathrm{Op}\,(K) \in L(C_0^\infty(\Omega_2), \mathcal{D}'(\Omega_1))$$

is injective. The next theorem asserts that it is also surjective:

Theorem 3.7 (the Schwartz kernel theorem). *If A is a continuous linear operator on $C_0^\infty(\Omega_2)$ into $\mathcal{D}'(\Omega_1)$, then there exists a unique distribution $K_A(x_1, x_2)$ in $\mathcal{D}'(\Omega_1 \times \Omega_2)$ such that $A = \mathrm{Op}\,(K)$.*

The distribution K_A is called the *kernel* of A. Formally we have the formula

$$A\psi(x_1) = \int_{\Omega_2} K_A(x_1, x_2)\, \psi(x_2)\, dx_2, \quad \psi \in C_0^\infty(\Omega_2).$$

Now we give some important examples of distributions kernels (see Example 3.2):

Example 3.6. (a) Riesz potentials: $\Omega_1 = \Omega_2 = \mathbf{R}^n$, $0 < \alpha < n$.

$$\begin{aligned}(-\Delta)^{-\alpha/2}u(x) &= R_\alpha * u(x) \\ &= \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-\alpha}}\, u(y)\, dy, \quad u \in C_0^\infty(\mathbf{R}^n).\end{aligned}$$

(b) Newtonian potentials: $\Omega_1 = \Omega_2 = \mathbf{R}^n$, $n \geq 3$.

$$\begin{aligned}(-\Delta)^{-1}u(x) &= N * u(x) \\ &= \frac{\Gamma((n-2)/2)}{4\pi^{n/2}} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-2}}\, u(y)\, dy, \quad u \in C_0^\infty(\mathbf{R}^n).\end{aligned}$$

(c) Bessel potentials: $\Omega_1 = \Omega_2 = \mathbf{R}^n$, $\alpha > 0$.

$$(I-\Delta)^{-\alpha/2}u(x) = G_\alpha * u(x) = \int_{\mathbf{R}^n} G_\alpha(x-y)\, u(y)\, dy, \quad u \in C_0^\infty(\mathbf{R}^n).$$

(d) Riesz operators: $\Omega_1 = \Omega_2 = \mathbf{R}^n$, $1 \leq j \leq n$.

$$\begin{aligned}Y_j u(x) &= R_j * u(x) \\ &= \sqrt{-1}\, \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\, \mathrm{v.\,p.} \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}\, u(y)\, dy, \quad u \in C_0^\infty(\mathbf{R}^n).\end{aligned}$$

(e) The Calderón–Zygmund integro-differential operator: $\Omega_1 = \Omega_2 = \mathbf{R}^n$.

$$\begin{aligned}(-\Delta)^{1/2}u(x) &= \frac{1}{\sqrt{-1}} \sum_{j=1}^n Y_j \left(\frac{\partial u}{\partial x_j}\right)(x) \\ &= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \sum_{j=1}^n \mathrm{v.\,p.} \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}}\, \frac{\partial u}{\partial y_j}(y)\, dy, \\ &\qquad u \in C_0^\infty(\mathbf{R}^n).\end{aligned}$$

Let Ω be an open subset of $\mathbf{R}^n$ and $m \in \mathbf{R}$. A *pseudo-differential operator* of order m on Ω is a Fourier integral operator of the form

$$Au(x) = \iint_{\Omega \times \mathbf{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y)\, dy d\xi, \quad u \in C_0^\infty(\Omega), \tag{3.4}$$

with some $a(x,y,\xi) \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbf{R}^n)$. In other words, a pseudo-differential operator of order m is a Fourier integral operator associated with the phase function $\varphi(x,y,\xi) = (x-y)\cdot\xi$ and some amplitude $a(x,y,\xi) \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbf{R}^n)$.

We let

$L^m_{\rho,\delta}(\Omega) =$ the set of all pseudo-differential operators of order m on Ω.

By applying Theorems 3.5 and 3.6 to our situation, we obtain the following three assertions:

(1) A pseudo-differential operator A maps $C_0^\infty(\Omega)$ continuously into $C^\infty(\Omega)$ and extends to a continuous linear operator $A : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$.
(2) The distribution kernel $K_A(x,y)$ of a pseudo-differential operator A satisfies the condition

$$\operatorname{sing\,supp} K_A \subset \{(x,x) : x \in \Omega\},$$

that is, the kernel K_A is smooth off the diagonal $\{(x,x) : x \in \Omega\}$ in $\Omega \times \Omega$.
(3) $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$, $u \in \mathcal{E}'(\Omega)$. In other words, Au is smooth whenever u is. This property is referred to as the *pseudo-local property.*

We set

$$L^{-\infty}(\Omega) = \bigcap_{m \in \mathbf{R}} L^m_{\rho,\delta}(\Omega).$$

The next theorem characterizes the class $L^{-\infty}(\Omega)$.

Theorem 3.8. *The following three conditions are equivalent:*

(i) $A \in L^{-\infty}(\Omega)$.
(ii) A *is written in the form (3.4) with some* $a \in S^{-\infty}(\Omega \times \Omega \times \mathbf{R}^n)$.
(iii) A *is a* regularizer*, or equivalently, its distribution kernel* $K_A(x,y)$ *is in the space* $C^\infty(\Omega \times \Omega)$.

We recall that a continuous linear operator $A : C_0^\infty(\Omega) \to \mathcal{D}'(\Omega)$ is said to be *properly supported* if the following two conditions are satisfied:

(a) For any compact subset K of Ω, there exists a compact subset K' of Ω such that

$$\operatorname{supp} v \subset K \implies \operatorname{supp} Av \subset K'.$$

(b) For any compact subset K' of Ω, there exists a compact subset $K \supset K'$ of Ω such that

$$\operatorname{supp} v \cap K = \emptyset \implies \operatorname{supp} Av \cap K' = \emptyset.$$

If A is properly supported, then it maps $C_0^\infty(\Omega)$ continuously into $\mathcal{E}'(\Omega)$ and extends to a continuous linear operator on $C^\infty(\Omega)$ into $\mathcal{D}'(\Omega)$.

The next theorem states that every pseudo-differential operator can be written as the sum of a properly supported operator and a regularizer.

Theorem 3.9. *If $A \in L^m_{\rho,\delta}(\Omega)$, then we have the decomposition*

$$A = A_0 + R,$$

where $A_0 \in L^m_{\rho,\delta}(\Omega)$ is properly supported and $R \in L^{-\infty}(\Omega)$.

If $p(x,\xi) \in S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n)$, then the operator $p(x,D)$, defined by the formula

$$p(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\, p(x,\xi)\, \widehat{u}(\xi)\, d\xi, \quad u \in C_0^\infty(\Omega), \tag{3.5}$$

is a pseudo-differential operator of order m on Ω, that is, $p(x,D) \in L^m_{\rho,\delta}(\Omega)$.

The next theorem asserts that every properly supported pseudo-differential operator can be reduced to the form (3.5).

Theorem 3.10. *If $A \in L^m_{\rho,\delta}(\Omega)$ is properly supported, then we have the assertion*

$$p(x,\xi) = e^{-ix\cdot\xi}\, A(e^{ix\cdot\xi}) \in S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n),$$

and

$$A = p(x,D).$$

Furthermore, if $a(x,y,\xi) \in S^m_{\rho,\delta}(\Omega \times \Omega \times \mathbf{R}^n)$ is an amplitude for A, then we have the following asymptotic expansion:

$$p(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha \left(a(x,y,\xi)\right)\Big|_{y=x}.$$

The function $p(x,\xi)$ is called the complete symbol *of A.*

We extend the notion of a complete symbol to the whole space $L^m_{\rho,\delta}(\Omega)$. If $A \in L^m_{\rho,\delta}(\Omega)$, then we choose a properly supported operator $A_0 \in L^m_{\rho,\delta}(\Omega)$ such that $A - A_0 \in L^{-\infty}(\Omega)$, and define

$$\begin{aligned}\sigma(A) = {} & \text{the equivalence class of the complete symbol of } A_0 \text{ in} \\ & S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n)/S^{-\infty}(\Omega \times \mathbf{R}^n).\end{aligned}$$

In view of Theorems 3.10 and 3.11, it follows that $\sigma(A)$ does not depend on the operator A_0 chosen. The equivalence class $\sigma(A)$ is called the *complete symbol* of A. It is easy to see that the mapping

$$L^m_{\rho,\delta}(\Omega) \ni A \longmapsto \sigma(A) \in S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n)/S^{-\infty}(\Omega \times \mathbf{R}^n)$$

induces an isomorphism

$$L^m_{\rho,\delta}(\Omega)/L^{-\infty}(\Omega) \longrightarrow S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n)/S^{-\infty}(\Omega \times \mathbf{R}^n).$$

We shall often identify the complete symbol $\sigma(A)$ with a representative in the class $S^m_{\rho,\delta}(\Omega \times \mathbf{R}^n)$ for notational convenience, and call any member of $\sigma(A)$ a complete symbol of A.

A pseudo-differential operator $A \in L^m_{1,0}(\Omega)$ is said to be *classical* if its complete symbol $\sigma(A)$ has a representative in the class $S^m_{\mathrm{cl}}(\Omega \times \mathbf{R}^n)$.

We let

$L^m_{\mathrm{cl}}(\Omega) =$ the set of all classical pseudo-differential operators of order m on Ω.

Then the mapping

$$L^m_{\mathrm{cl}}(\Omega) \ni A \longmapsto \sigma(A) \in S^m_{\mathrm{cl}}(\Omega \times \mathbf{R}^n)/S^{-\infty}(\Omega \times \mathbf{R}^n)$$

induces an isomorphism

$$L^m_{\mathrm{cl}}(\Omega)/L^{-\infty}(\Omega) \longrightarrow S^m_{\mathrm{cl}}(\Omega \times \mathbf{R}^n)/S^{-\infty}(\Omega \times \mathbf{R}^n).$$

Also we have the assertion

$$L^{-\infty}(\Omega) = \bigcap_{m \in \mathbf{R}} L^m_{\mathrm{cl}}(\Omega).$$

If $A \in L^m_{\mathrm{cl}}(\Omega)$, then the principal part of $\sigma(A)$ has a canonical representative $\sigma_A(x,\xi) \in C^\infty(\Omega \times (\mathbf{R}^n \setminus \{0\}))$ which is positively homogeneous of degree m in the variable ξ. The function $\sigma_A(x,\xi)$ is called the *homogeneous principal symbol* of A.

The next two theorems assert that the class of pseudo-differential operators forms an algebra closed under the operations of composition of operators and taking the transpose or adjoint of an operator.

Theorem 3.11. *If $A \in L^m_{\rho,\delta}(\Omega)$, then its transpose A' and its adjoint A^* are both in $L^m_{\rho,\delta}(\Omega)$, and the complete symbols $\sigma(A')$ and $\sigma(A^*)$ have respectively the following asymptotic expansions:*

$$\sigma(A')(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \left(\sigma(A)(x,-\xi)\right),$$

$$\sigma(A^*)(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \left(\overline{\sigma(A)(x,\xi)}\right).$$

Theorem 3.12. *If $A \in L^{m'}_{\rho',\delta'}(\Omega)$ and $B \in L^{m''}_{\rho'',\delta''}(\Omega)$ where $0 \leq \delta' < \rho'' \leq 1$ and if one of them is properly supported, then the composition AB is in*

$L^{m'+m''}_{\rho,\delta}(\Omega)$ *with* $\rho = \min(\rho', \rho'')$, $\delta = \max(\delta', \delta'')$, *and we have the following asymptotic expansion:*

$$\sigma(AB)(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \left(\sigma(A)(x,\xi)\right) \cdot D_x^\alpha \left(\sigma(B)(x,\xi)\right).$$

A pseudo-differential operator $A \in L^m_{\rho,\delta}(\Omega)$ is said to be *elliptic* of order m if its complete symbol $\sigma(A)$ is elliptic of order m. In view of Theorem 3.2, it follows that a classical pseudo-differential operator $A \in L^m_{\mathrm{cl}}(\Omega)$ is elliptic if and only if its homogeneous principal symbol $\sigma_A(x,\xi)$ does not vanish on the space $\Omega \times (\mathbf{R}^n \setminus \{0\})$.

The next theorem states that elliptic operators are the "invertible" elements in the algebra of pseudo-differential operators.

Theorem 3.13. *An operator* $A \in L^m_{\rho,\delta}(\Omega)$ *is elliptic if and only if there exists a properly supported operator* $B \in L^{-m}_{\rho,\delta}(\Omega)$ *such that:*

$$\begin{cases} AB \equiv I \mod L^{-\infty}(\Omega), \\ BA \equiv I \mod L^{-\infty}(\Omega). \end{cases}$$

Such an operator B is called a *parametrix* for A. In other words, a parametrix for A is a two-sided inverse of A modulo $L^{-\infty}(\Omega)$. We observe that a parametrix is unique modulo $L^{-\infty}(\Omega)$.

The next theorem proves the invariance of pseudo-differential operators under change of coordinates.

Theorem 3.14. *Let* Ω_1 *and* Ω_2 *be two open subsets of* $\mathbf{R}^n$ *and let* $\chi : \Omega_1 \to \Omega_2$ *be a* C^∞ *diffeomorphism. If* $A \in L^m_{\rho,\delta}(\Omega_1)$*, where* $1-\rho \leq \delta < \rho \leq 1$*, then the mapping*

$$\begin{aligned} A_\chi : \ C_0^\infty(\Omega_2) &\longrightarrow C^\infty(\Omega_2) \\ v &\longmapsto A(v \circ \chi) \circ \chi^{-1} \end{aligned}$$

is in $L^m_{\rho,\delta}(\Omega_2)$*, and we have the asymptotic expansion*

$$\sigma(A_\chi)(y,\eta) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \left(\partial_\xi^\alpha \sigma(A)\right)(x, {}^t\chi'(x) \cdot \eta) \cdot D_z^\alpha \left(e^{ir(x,z,\eta)}\right)\Big|_{z=x} \tag{3.6}$$

with

$$r(x,z,\eta) = \langle \chi(z) - \chi(x) - \chi'(x) \cdot (z-x), \eta \rangle.$$

Here $x = \chi^{-1}(y)$*,* $\chi'(x)$ *is the derivative of* χ *at* x *and* ${}^t\chi'(x)$ *is its transpose.*

Remark 3.1. Formula (3.6) shows that

$$\sigma(A_\chi)(y,\eta) \equiv \sigma(A)\left(x, {}^t\chi'(x) \cdot \eta\right) \mod S^{m-(\rho-\delta)}_{\rho,\delta}.$$

Note that the mapping

$$\Omega_2 \times \mathbf{R}^n \ni (y, \eta) \longmapsto \left(x, {}^t\chi'(x) \cdot \eta\right) \in \Omega_1 \times \mathbf{R}^n$$

is just a transition map of the cotangent bundle $T^*(\mathbf{R}^n)$. This implies that the principal symbol $\sigma_m(A)$ of $A \in L^m_{\rho,\delta}(\mathbf{R}^n)$ can be invariantly defined on $T^*(\mathbf{R}^n)$ when $1 - \rho \leq \delta < \rho \leq 1$.

The situation may be represented by the following diagram:

$$\begin{array}{ccc} C_0^\infty(\Omega_1) & \xrightarrow{\ A\ } & C^\infty(\Omega_1) \\ \chi^* \Big\uparrow & & \Big\downarrow \chi_* \\ C_0^\infty(\Omega_2) & \xrightarrow[A_\chi]{} & C^\infty(\Omega_2) \end{array}$$

Here $\chi^* v = v \circ \chi$ is the pull-back of v by χ and $\chi_* u = u \circ \chi^{-1}$ is the push-forward of u by χ, respectively.

A differential operator of order m with smooth coefficients on Ω is continuous on $H^{s,p}_{\text{loc}}(\Omega)$ (resp. $B^{s,p}_{\text{loc}}(\Omega)$) into $H^{s-m,p}_{\text{loc}}(\Omega)$ (resp. $B^{s-m,p}_{\text{loc}}(\Omega)$) for all $s \in \mathbf{R}$. This result extends to pseudo-differential operators:

Theorem 3.15 (the Besov space boundedness theorem). *Every properly supported operator* $A \in L^m_{1,\delta}(\Omega)$, $0 \leq \delta < 1$, *extends to a continuous linear operator*

$$A : H^{s,p}_{\text{loc}}(\Omega) \longrightarrow H^{s-m,p}_{\text{loc}}(\Omega)$$

for all $s \in \mathbf{R}$ *and* $1 < p < \infty$, *and also it extends to a continuous linear operator*

$$A : B^{s,p}_{\text{loc}}(\Omega) \longrightarrow B^{s-m,p}_{\text{loc}}(\Omega)$$

for all $s \in \mathbf{R}$ *and* $1 \leq p \leq \infty$.

For a proof of Theorem 3.15, the reader might refer to Bourdaud [Bo, Theorem 1] (see also [Ta5, Appendix A]).

Now we define the concept of a pseudo-differential operator on a manifold, and transfer all the machinery of pseudo-differential operators to manifolds. Let M be an n-dimensional *compact* smooth manifold without boundary. Theorem 3.14 leads us to the following:

Definition 3.1. Let $1 - \rho \leq \delta < \rho \leq 1$. A continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ is called a *pseudo-differential operator* of order $m \in \mathbf{R}$ if it satisfies the following two conditions:

(i) The distribution kernel $K_A(x, y)$ of A is smooth off the diagonal $\{(x, x) : x \in M\}$ in $M \times M$.
(ii) For any chart (U, χ) on M, the mapping

$$\begin{aligned} A_\chi : \ & C_0^\infty(\chi(U)) \longrightarrow C^\infty(\chi(U)) \\ & u \longmapsto A(u \circ \chi) \circ \chi^{-1} \end{aligned}$$

belongs to the class $L^m_{\rho,\delta}(\chi(U))$.

We let

$$L^m_{\rho,\delta}(M) = \text{the set of all pseudo-differential operators of order } m \text{ on } M,$$

and set

$$L^{-\infty}(M) = \bigcap_{m\in\mathbf{R}} L^m_{\rho,\delta}(M).$$

Some results about pseudo-differential operators on $\mathbf{R}^n$ stated above are also true for pseudo-differential operators on M. In fact, pseudo-differential operators on M are defined to be locally pseudo-differential operators on $\mathbf{R}^n$.

For example, we have the following five assertions:

(1) A pseudo-differential operator A extends to a continuous linear operator $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$.
(2) $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$, $u \in \mathcal{D}'(M)$.
(3) A continuous linear operator $A : C^\infty(M) \to \mathcal{D}'(M)$ is a —em regularizer if and only if it is in the class $L^{-\infty}(M)$.
(4) The class $L^m_{\rho,\delta}(M)$ is stable under the operations of composition of operators and taking the transpose or adjoint of an operator.
(5) (**The Besov space boundedness theorem**) A pseudo-differential operator $A \in L^m_{1,\delta}(M)$, $0 \leq \delta < 1$, extends to a continuous linear operator

$$A : H^{s,p}(M) \longrightarrow H^{s-m,p}(M)$$

for all $s \in \mathbf{R}$ and $1 < p < \infty$, and also it extends to a continuous linear operator

$$A : B^{s,p}(M) \longrightarrow B^{s-m,p}(M)$$

for all $s \in \mathbf{R}$ and $1 \leq p \leq \infty$.

A pseudo-differential operator $A \in L^m_{1,0}(M)$ is said to be *classical* if, for any chart (U, χ) on M, the mapping $A_\chi : C_0^\infty(\chi(U)) \to C^\infty(\chi(U))$ belongs to the class $L^m_{\mathrm{cl}}(\chi(U))$.

We let

$$L^m_{\mathrm{cl}}(M) = \text{the set of all classical pseudo-differential operators of order } m \text{ on } M.$$

We observe that

$$L^{-\infty}(M) = \bigcap_{m\in\mathbf{R}} L^m_{\mathrm{cl}}(M).$$

Let $A \in L^m_{\mathrm{cl}}(M)$. If (U, χ) is a chart on M, there is associated a homogeneous principal symbol $\sigma_{A_\chi} \in C^\infty(\chi(U) \times (\mathbf{R}^n \setminus \{0\}))$. In view of Remark 3.1, by smoothly patching together the functions σ_{A_χ} we can obtain a smooth function $\sigma_A(x, \xi)$ on $T^*(M) \setminus \{0\} = \{(x, \xi) \in T^*(M) : \xi \neq 0\}$, which is positively homogeneous of degree m in the variable ξ. The function $\sigma_A(x, \xi)$ is called the *homogeneous principal symbol* of A.

A classical pseudo-differential operator $A \in L^m_{\mathrm{cl}}(M)$ is said to be *elliptic* of order m if its homogeneous principal symbol $\sigma_A(x,\xi)$ does not vanish on the bundle $T^*(M) \setminus \{0\}$ of non-zero cotangent vectors.

Then we have the following assertion:

(6) An operator $A \in L^m_{\mathrm{cl}}(M)$ is elliptic if and only if there exists a parametrix $B \in L^{-m}_{\mathrm{cl}}(M)$ for A:

$$\begin{cases} AB \equiv I \bmod L^{-\infty}(M), \\ BA \equiv I \bmod L^{-\infty}(M). \end{cases}$$

Let Ω be an open subset of $\mathbf{R}^n$. A properly supported pseudo-differential operator A on Ω is said to be *hypoelliptic* if it satisfies the condition

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Au, \quad u \in \mathcal{D}'(\Omega).$$

For example, Theorem 3.13 asserts that elliptic operators are hypoelliptic. We remark that this notion may be transferred to manifolds.

The following criterion for hypoellipticity is due to Hörmander (cf. [Ho2, Theorem 4.2]):

Theorem 3.16. *Let $A = p(x,D) \in L^m_{\rho,\delta}(\Omega)$ be properly supported. Assume that, for any compact $K \subset \Omega$ and any multi-indices α, β, there exist positive constants $C_{K,\alpha,\beta}$, C_K and a real number μ such that we have, for all $x \in K$ and $|\xi| \geq C_K$,*

$$\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \leq C_{K,\alpha,\beta} |p(x,\xi)| \left(1+|\xi|\right)^{-\rho|\alpha|+\delta|\beta|}, \tag{3.7a}$$

$$|p(x,\xi)|^{-1} \leq C_K \left(1+|\xi|\right)^{\mu}. \tag{3.7b}$$

Then there exists a parametrix $B \in L^{\mu}_{\rho,\delta}(\Omega)$ *for A.*

4

L^p Approach to Elliptic Boundary Value Problems

In this chapter we study elliptic boundary value problems in the framework of L^p-spaces, by using the L^p theory of pseudo-differential operators. For more thorough treatments of this subject, the reader might refer to Hörmander [Ho1], Seeley [Se2], Taylor [Ty, Chapter XI] and also Taira [Ta2, Chapter 8] (L^2-version).

4.1 The Dirichlet Problem

In this section we shall consider the Dirichlet problem in the framework of Sobolev spaces of L^p type. This is a generalization of the classical potential approach to the Dirichlet problem.

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an n-dimensional, compact smooth manifold with boundary. We may assume that $\overline{\Omega}$ is the closure of a relatively compact open subset Ω of an n-dimensional, compact smooth manifold M without boundary in which Ω has a smooth boundary Γ. This manifold M is the *double* of Ω (see Figure 4.1).

We let

$$A = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a second-order, *elliptic* differential operator with real coefficients such that:

(1) $a^{ij} \in C^\infty(M)$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in M$, $1 \leq i, j \leq n$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq a_0 |\xi|^2 \quad \text{on } T^*(M).$$

Here $T^*(M)$ is the cotangent bundle of the double M.

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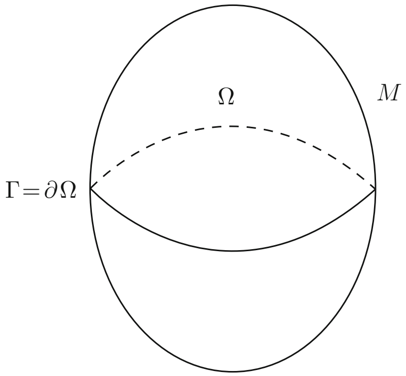

Fig. 4.1.

(2) $b^i \in C^\infty(M)$ for all $1 \leq i \leq n$.
(3) $c \in C^\infty(M)$ and $c(x) \leq 0$ in M.

Furthermore, for simplicity, we assume that:

$$\text{The function } c(x) \text{ does not vanish } \textit{identically} \text{ on the double } M. \tag{4.1}$$

The next theorem states the existence of a *volume potential* for A, which plays the same role for A as the Newtonian potential plays for the Laplacian (cf. [Se1, Theorem 1] and [Ta2, Theorem 8.2.1]):

Theorem 4.1. *(i) The operator $A : C^\infty(M) \to C^\infty(M)$ is bijective, and its inverse Q is a classical* elliptic *pseudo-differential operator of order -2 on M.*
(ii) The operators A and Q extend respectively to isomorphisms

$$A : H^{s,p}(M) \longrightarrow H^{s-2,p}(M),$$
$$Q : H^{s-2,p}(M) \longrightarrow H^{s,p}(M)$$

for all $s \in \mathbf{R}$, which are still inverses of each other.

Next we construct a *surface potential* for A, which is a generalization of the classical Poisson kernel for the Laplacian.

We let

$$Kv = Q(v \otimes \delta)|_\Gamma, \quad v \in C^\infty(\Gamma).$$

Here $v(x') \otimes \delta(t)$ is a distribution on M defined by the formula

$$\langle v \otimes \delta, \varphi \cdot \mu \rangle = \langle v, \varphi(\cdot, 0) \cdot \omega \rangle, \quad \varphi(x', t) \in C^\infty(M),$$

where μ is a strictly positive density on M and ω is a strictly positive density on $\Gamma = \{t = 0\}$, respectively.

Then we have the following (cf. [Ta2, Theorem 8.2.2]):

Theorem 4.2. *(i) The operator K is a classical* elliptic *pseudo-differential operator of order -1 on Γ.*
(ii) The operator $K : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is bijective, and its inverse L is a classical elliptic pseudo-differential operator of first order on Γ. Furthermore, the operators K and L extend respectively to isomorphisms

$$K : B^{\sigma,p}(\Gamma) \longrightarrow B^{\sigma+1,p}(\Gamma),$$
$$L : B^{\sigma+1,p}(\Gamma) \longrightarrow B^{\sigma,p}(\Gamma)$$

for all $\sigma \in \mathbf{R}$, which are still inverses of each other.

Now we let

$$P\varphi = Q(L\varphi \otimes \delta)|_\Omega, \quad \varphi \in C^\infty(\Gamma).$$

Then the operator P maps $C^\infty(\Gamma)$ continuously into $C^\infty(\overline{\Omega})$, and extends to a continuous linear operator

$$P : B^{s-1/p,p}(\Gamma) \longrightarrow H^{s,p}(\Omega)$$

for all $s \in \mathbf{R}$. Furthermore, we have, for all $\varphi \in B^{s-1/p,p}(\Gamma)$,

$$\begin{cases} AP\varphi = AQ(L\varphi \otimes \delta)|_\Omega = (L\varphi \otimes \delta)|_\Omega = 0 & \text{in } \Omega, \\ P\varphi|_\Gamma = KL\varphi = \varphi & \text{on } \Gamma. \end{cases}$$

The operator P is called the *Poisson operator*.

We let

$$N(A, s, p) = \{u \in H^{s,p}(\Omega) : Au = 0 \text{ in } \Omega\}, \quad s \in \mathbf{R}.$$

Since the injection $H^{s,p}(\Omega) \to \mathcal{D}'(\Omega)$ is continuous, it follows that the null space $N(A, s, p)$ is a closed subspace of $H^{s,p}(\Omega)$; hence it is a Banach space.

Then we have the following fundamental result (cf. [Se2, Theorems 5 and 6]):

Theorem 4.3. *The Poisson operator P maps the Besov space $B^{s-1/p,p}(\Gamma)$ isomorphically onto the null space $N(A, s, p)$ for all $s \in \mathbf{R}$.*

By combining Theorems 4.1 and 4.3, we can obtain the following existence and uniqueness theorem for the Dirichlet problem (cf. [ADN]):

Theorem 4.4. *Let $s \geq 2$. The Dirichlet problem*

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma \end{cases} \tag{4.2}$$

has a unique solution $u(x)$ in $H^{s,p}(\Omega)$ for any $f \in H^{s-2,p}(\Omega)$ and any $\varphi \in B^{s-1/p,p}(\Gamma)$.

Furthermore, we can prove the following existence and uniqueness theorem for the Neumann problem (cf. [ADN]):

Theorem 4.5. *Let $s \geq 2$. The Neumann problem*

$$\begin{cases} Au = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \varphi & \text{on } \Gamma \end{cases} \tag{4.3}$$

has a unique solution $u(x)$ in $H^{s,p}(\Omega)$ for any $f \in H^{s-2,p}(\Omega)$ and any $\varphi \in B^{s-1-1/p,p}(\Gamma)$. Here $\mathbf{n}$ is the unit interior normal to the boundary Γ.

By Theorem 4.5, we can introduce a linear operator

$$G_N : H^{s-2,p}(\Omega) \longrightarrow H^{s,p}(\Omega)$$

as follows: For any $f \in H^{s-2,p}(\Omega)$, the function $G_N f \in H^{s,p}(\Omega)$ is the unique solution of the problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma. \end{cases}$$

The operator G_N is called the *Green operator* for the Neumann problem.

4.2 Formulation of a Boundary Value Problem

If $u \in H^{2,p}(\Omega) = W^{2,p}(\Omega)$, we can define its traces $\gamma_0 u$ and $\gamma_1 u$ respectively by the formulas

$$\begin{cases} \gamma_0 u = u|_\Gamma, \\ \gamma_1 u = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_\Gamma, \end{cases}$$

and let

$$\boldsymbol{\gamma} u = \{\gamma_0 u, \gamma_1 u\}.$$

Then we have the following (cf. [St1]):

Theorem 4.6 (the trace theorem). *The trace map*

$$\boldsymbol{\gamma} : H^{2,p}(\Omega) \longrightarrow B^{2-1/p,p}(\Gamma) \oplus B^{1-1/p,p}(\Gamma)$$

is continuous and surjective for all $1 < p < \infty$.

We define a boundary condition

$$Bu := a(x') \frac{\partial u}{\partial \mathbf{n}} + b(x')u \bigg|_\Gamma = a(x')\gamma_1 u + b(x')\gamma_0 u, \quad u \in H^{2,p}(\Omega),$$

where $a(x')$ and $b(x')$ are real-valued, smooth functions on Γ.

Then we have the following:

Proposition 4.7. *The mapping*

$$B : H^{2,p}(\Omega) \longrightarrow B^{1-1/p,p}(\Gamma)$$

is continuous for all $1 < p < \infty$.

Now we can formulate our boundary value problem for (A, B) as follows: Given functions $f \in L^p(\Omega)$ and $\varphi \in B^{2-1/p,p}(\Gamma)$, find a function $u \in H^{2,p}(\Omega)$ such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = \varphi & \text{on } \Gamma. \end{cases} \tag{4.4}$$

4.3 Reduction to the Boundary

In this section, by using the operators P and G_N we shall show that problem (4.4) can be reduced to the study of a pseudo-differential operator on the boundary.

First, we remark that every function $u(x)$ in $H^{2,p}(\Omega)$ can be written in the following form:

$$u(x) = v(x) + w(x), \tag{4.5}$$

where

$$\begin{cases} v = G_N(Au) \in H^{2,p}(\Omega), \\ w = u - v \in N(A, 2, p) = \left\{ z \in H^{2,p}(\Omega) : Az = 0 \text{ in } \Omega \right\}. \end{cases}$$

Since the operator $G_N : L^p(\Omega) \to H^{2,p}(\Omega)$ is continuous, it follows that the decomposition (4.5) is continuous; more precisely, we have the inequalities

$$\|v\|_{2,p} \leq C\|Au\|_p \leq C\|u\|_{2,p};$$
$$\|w\|_{2,p} \leq \|u\|_{2,p} + \|v\|_{2,p} \leq C\|u\|_{2,p}.$$

Here the letter C denotes a generic positive constant.

Now we assume that $u \in H^{2,p}(\Omega)$ is a solution of the boundary value problem (4.4)

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = \varphi & \text{on } \Gamma. \end{cases}$$

Then, by virtue of the decomposition (4.5) of $u(x)$ this is equivalent to saying that $w \in H^{2,p}(\Omega)$ is a solution of the boundary value problem

$$\begin{cases} Aw = 0 & \text{in } \Omega, \\ Bw = \varphi - Bv & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Here $v(x) = G_N f \in H^{2,p}(\Omega)$ and $w(x) = u(x) - v(x)$. However, Theorem 4.3 asserts that the spaces $N(A, 2, p)$ and $B^{2-1/p,p}(\Gamma)$ are isomorphic in such a way that:

$$N(A, 2, p) \xrightarrow{\gamma_0} B^{2-1/p,p}(\Gamma).$$
$$N(A, 2, p) \xleftarrow[P]{} B^{2-1/p,p}(\Gamma).$$

Therefore, we find that $w \in H^{2,p}(\Omega)$ is a solution of problem (4.6) if and only if $\psi(x') \in B^{2-1/p,p}(\Gamma)$ is a solution of the equation

$$BP\psi = \varphi - Bv \quad \text{on } \Gamma. \tag{4.7}$$

Here $\psi = \gamma_0 w$, or equivalently, $w = P\psi$.

Summing up, we obtain the following:

Proposition 4.8. *For functions $f \in L^p(\Omega)$ and $\varphi \in B^{2-1/p,p}(\Gamma)$, there exists a solution $u \in H^{2,p}(\Omega)$ of problem (4.4) if and only if there exists a solution $\psi \in B^{2-1/p,p}(\Gamma)$ of equation (4.7). Furthermore, the solutions $u(x)$ and $\psi(x')$ are related as follows:*

$$u = G_N f + P\psi.$$

We remark that equation (4.7) is a generalization of the classical *Fredholm integral equation.*

We let

$$\begin{aligned} T: \ C^\infty(\Gamma) &\longrightarrow C^\infty(\Gamma) \\ \varphi &\longmapsto BP\varphi. \end{aligned}$$

Then we have, by condition (4.2),

$$T = a(x')\Pi + b(x'),$$

where

$$\Pi\varphi = \gamma_1 P\varphi = \frac{\partial}{\partial \mathbf{n}}(P\varphi)\bigg|_\Gamma, \quad \varphi \in C^\infty(\Gamma).$$

It is known (cf. [Ho1], [Se2]) that the operator Π is a classical pseudo-differential operator of first order on Γ; hence the operator T is a classical pseudo-differential operator of first order on the boundary Γ.

Consequently, Proposition 4.8 asserts that problem (4.4) can be reduced to the study of the first-order pseudo-differential operator T on the boundary Γ. We shall formulate this fact more precisely in terms of functional analysis.

First, we remark that the operator $T : C^\infty(\Gamma) \to C^\infty(\Gamma)$ extends to a continuous linear operator

$$T : B^{s,p}(\Gamma) \longrightarrow B^{s-1,p}(\Gamma), \quad s \in \mathbf{R}.$$

Then we have the formula

$$T\varphi = BP\varphi, \quad \varphi \in B^{2-1/p,p}(\Gamma),$$

since the Poisson operator

$$P : B^{2-1/p,p}(\Gamma) \longrightarrow N(A, 2, p)$$

and the boundary operator

$$B : H^{2,p}(\Omega) \longrightarrow B^{1-1/p,p}(\Gamma)$$

are both continuous.

We associate with problem (4.4) a linear operator

$$\mathcal{A} : H^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\Gamma)$$

as follows.

(a) The domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is the space

$$\mathcal{D}(\mathcal{A}) = \left\{ u \in H^{2,p}(\Omega) = W^{2,p}(\Omega) : Bu \in B^{2-1/p,p}(\Gamma) \right\}.$$

(b) $\mathcal{A}u = \{Au, Bu\}$, $u \in \mathcal{D}(\mathcal{A})$.

Note that the space $B^{2-1/p,p}(\Gamma)$ is a right boundary space associated with the Dirichlet condition: $a(x') \equiv 0$ and $b(x') \equiv 1$ on Γ.

Since the operators $A : H^{2,p}(\Omega) \to L^p(\Omega)$ and $B : H^{2,p}(\Omega) \to B^{1-1/p,p}(\Gamma)$ are both continuous, it follows that $\mathcal{A}$ is a closed operator. Furthermore, the operator $\mathcal{A}$ is densely defined, since the domain $\mathcal{D}(\mathcal{A})$ contains the space $C^\infty(\overline{\Omega})$.

Similarly, we associate with equation (4.7) a linear operator

$$\mathcal{T} : B^{2-1/p,p}(\Gamma) \longrightarrow B^{2-1/p,p}(\Gamma)$$

as follows.

(α) The domain $\mathcal{D}(\mathcal{T})$ of $\mathcal{T}$ is the space

$$\mathcal{D}(\mathcal{T}) = \left\{ \varphi \in B^{2-1/p,p}(\Gamma) : T\varphi \in B^{2-1/p,p}(\Gamma) \right\}.$$

(β) $\mathcal{T}\varphi = T\varphi$, $\varphi \in \mathcal{D}(\mathcal{T})$.

Then the operator $\mathcal{T}$ is a densely defined, closed operator, since the operator $T : B^{2-1/p,p}(\Gamma) \to B^{1-1/p,p}(\Gamma)$ is continuous and since the domain $\mathcal{D}(\mathcal{T})$ contains the space $C^\infty(\Gamma)$.

The next theorem states that $\mathcal{A}$ has regularity property if and only if $\mathcal{T}$ has.

Theorem 4.9. *The following two conditions are equivalent:*

$$u \in L^p(\Omega),\ Au \in L^p(\Omega) \text{ and } Bu \in B^{2-1/p,p}(\Gamma) \Longrightarrow u \in H^{2,p}(\Omega). \quad (4.8)$$

$$\varphi \in B^{-1/p,p}(\Gamma) \text{ and } T\varphi \in B^{2-1/p,p}(\Gamma) \Longrightarrow \varphi \in B^{2-1/p,p}(\Gamma). \quad (4.9)$$

Proof. (i) First, we show that assertion (4.8) implies assertion (4.9). To do this, assume that

$$\varphi \in B^{-1/p,p}(\Gamma) \quad \text{and} \quad T\varphi \in B^{2-1/p,p}(\Gamma).$$

Then, by letting $u = P\varphi$ we obtain that

$$u \in L^p(\Omega),\ Au = 0 \quad \text{and} \quad Bu = T\varphi \in B^{2-1/p,p}(\Gamma).$$

Hence it follows from condition (4.8) that

$$u \in H^{2,p}(\Omega),$$

so that, by Theorem 4.6,

$$\varphi = \gamma_0 u \in B^{2-1/p,p}(\Gamma).$$

(ii) Conversely, we show that assertion (4.9) implies estimate (4.8). To do this, assume that

$$u \in L^p(\Omega), \ Au \in L^p(\Omega) \quad \text{and} \quad Bu \in B^{2-1/p,p}(\Gamma).$$

Then the distribution $u(x)$ can be decomposed as follows:

$$u(x) = v(x) + w(x),$$

where

$$\begin{cases} v = G_N(Au) \in H^{2,p}(\Omega), \\ w = u - v \in N(A, 0, p) = \{z \in L^p(\Omega) : Az = 0 \text{ in } \Omega\}. \end{cases}$$

Moreover, Theorem 4.3 asserts that the distribution $w(x)$ can be written in the form

$$w = P\varphi, \quad \varphi = \gamma_0 w \in B^{-1/p,p}(\Gamma).$$

Hence we have, by Theorem 4.6,

$$T\varphi = BP\varphi = Bw = Bu - Bv = Bu - b(x')\gamma_0 v \in B^{2-1/p,p}(\Gamma),$$

since $\gamma_1 v = 0$. Thus it follows from condition (4.9) that

$$\varphi \in B^{2-1/p,p}(\Gamma),$$

so that again, by Theorem 4.3,

$$w = P\varphi \in H^{2,p}(\Omega).$$

This proves that

$$u = v + w \in H^{2,p}(\Omega).$$

The proof of Theorem 4.9 is complete. □

The next theorem states that *a priori* estimates for $\mathcal{A}$ are entirely equivalent to corresponding *a priori* estimates for $\mathcal{T}$.

Theorem 4.10. *The following two estimates are equivalent:*

$$\|u\|_{2,p} \leq C \left(\|Au\|_p + |Bu|_{2-1/p,p} + \|u\|_p \right), \quad u \in \mathcal{D}(\mathcal{A}). \tag{4.10}$$

$$|\varphi|_{2-1/p,p} \leq C \left(|\mathcal{T}\varphi|_{2-1/p,p} + |\varphi|_{-1/p,p} \right), \quad \varphi \in \mathcal{D}(\mathcal{T}). \tag{4.11}$$

Here and in the following the letter C denotes a generic positive constant.

Proof. (i) First, we show that estimate (4.10) implies estimate (4.11).

By taking $u = P\varphi$ with $\varphi \in \mathcal{D}(\mathcal{T})$ in estimate (4.10), we obtain that

$$\|P\varphi\|_{2,p} \leq C \left(|\mathcal{T}\varphi|_{2-1/p,p} + \|P\varphi\|_p \right). \tag{4.12}$$

However, Theorem 4.3 asserts that the Poisson operator P maps the Besov space $B^{s-1/p,p}(\varGamma)$ isomorphically onto the null space $N(A, s, p)$ for all $s \in \mathbf{R}$. Thus the desired estimate (4.11) follows from estimate (4.12).

(ii) Conversely, we show that estimate (4.11) implies estimate (4.10).

To do this, we express a function $u \in \mathcal{D}(\mathcal{A})$ in the form

$$u(x) = v(x) + w(x),$$

where

$$\begin{cases} v = G_N(Au) \in H^{2,p}(\varOmega), \\ w = u - v \in N(A, 2, p) = \left\{ z \in H^{2,p}(\varOmega) : Az = 0 \text{ in } \varOmega \right\}. \end{cases}$$

Then we have, by Theorem 4.5 with $s := 2$,

$$\|v\|_{2,p} = \|G_N(Au)\|_{2,p} \leq C\|Au\|_p. \tag{4.13}$$

Furthermore, by applying estimate (4.11) to the distribution $\gamma_0 w$ we obtain that

$$\begin{aligned} |\gamma_0 w|_{2-1/p,p} &\leq C \left(|\mathcal{T}(\gamma_0 w)|_{2-1/p,p} + |\gamma_0 w|_{-1/p,p} \right) \\ &= C \left(|Bw|_{2-1/p,p} + |\gamma_0 w|_{-1/p,p} \right) \\ &\leq C \left(|Bu|_{2-1/p,p} + |Bv|_{2-1/p,p} + |\gamma_0 w|_{-1/p,p} \right). \end{aligned}$$

In view of Theorem 4.3, this proves that

$$\begin{aligned} \|w\|_{2,p} &\leq C \left(|Bu|_{2-1/p,p} + |Bv|_{2-1/p,p} + \|w\|_p \right) \\ &\leq C \left(|Bu|_{2-1/p,p} + |Bv|_{2-1/p,p} + \|u\|_p + \|v\|_p \right) \\ &\leq C \left(|Bu|_{2-1/p,p} + |Bv|_{2-1/p,p} + \|u\|_p + \|v\|_{2,p} \right). \end{aligned} \tag{4.14}$$

However, since $\gamma_1 v = 0$, it follows from an application of Theorem 4.6 that

$$|Bv|_{2-1/p,p} = |b(x')\gamma_0 v|_{2-1/p,p} \leq C\|v\|_{2,p}. \tag{4.15}$$

Thus, by carrying estimates (4.13) and (4.15) into estimate (4.14) we obtain that

$$\|w\|_{2,p} \leq C \left(\|Au\|_p + |Bu|_{2-1/p,p} + \|u\|_p \right). \tag{4.16}$$

Therefore, the desired estimate (4.10) follows from estimates (4.13) and (4.16), since $u(x) = v(x) + w(x)$.

The proof of Theorem 4.10 is complete. □

5

Proof of Theorem 1.1

This chapter is devoted to the proof of Theorem 1.1. The idea of our proof is stated as follows. First, we reduce the study of the boundary value problem

$$\begin{cases} (A-\lambda)u = f & \text{in } D, \\ Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = \varphi & \text{on } \partial D \end{cases} \tag{1.1}$$

to that of a first-order pseudo-differential operator $T(\lambda) = LP(\lambda)$ on the boundary ∂D, just as in Section 4.3. Then we prove that conditions (A) and (B) are sufficient for the validity of the *a priori* estimate

$$\|u\|_{2,p} \leq C(\lambda)\left(\|f\|_p + |\varphi|_{2-1/p,p} + \|u\|_p\right). \tag{1.2}$$

More precisely, we construct a *parametrix* $S(\lambda)$ for $T(\lambda)$ in the Hörmander class $L^0_{1,1/2}(\partial D)$ (Lemma 5.2), and apply the Besov-space boundedness theorem (Theorem 3.15) to $S(\lambda)$ to obtain the desired estimate (1.2) (Lemma 5.1).

5.1 Boundary Value Problem with Spectral Parameter

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$ with smooth boundary ∂D. Its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We may assume that $\overline{D}$ is the closure of a relatively compact open subset D of an N-dimensional, compact smooth manifold $\widehat{D}$ without boundary in which D has a smooth boundary ∂D. This manifold $\widehat{D}$ is the *double* of D (see Figure 5.1).

We let

$$A = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial}{\partial x_i} + c(x)$$

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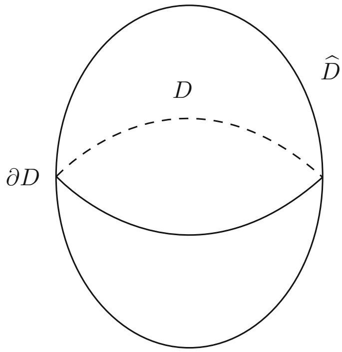

Fig. 5.1.

be a second-order, *elliptic* differential operator with real coefficients such that:

(1) $a^{ij} \in C^\infty(\widehat{D})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \widehat{D}$, $1 \leq i, j \leq N$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{on } T^*(\widehat{D}),$$

where $T^*(\widehat{D})$ is the cotangent bundle of the double $\widehat{D}$.

(2) $b^i \in C^\infty(\widehat{D})$ for all $1 \leq i \leq N$.

(3) $c \in C^\infty(\widehat{D})$ and $c(x) \leq 0$ in D.

In this chapter we consider the elliptic boundary value problem with spectral parameter

$$\begin{cases} (A - \lambda)u = f & \text{in } D, \\ Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = \varphi & \text{on } \partial D. \end{cases} \tag{1.1}$$

Here we recall that:

(4) λ is a *complex* parameter.

(5) $\mu(x')$ and $\gamma(x')$ are real-valued, smooth functions on the boundary ∂D.

(6) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D (see Figure 1.1).

The purpose of this chapter is to prove Theorem 1.1. More precisely, we prove the *a priori* estimate

$$\|u\|_{2,p} \leq C(\lambda) \left(\|(A - \lambda)u\|_p + |Lu|_{2-1/p,p} + \|u\|_p \right), \tag{1.2}$$

provided that the following two conditions (A) and (B) are satisfied:

(A) $\mu(x') \geq 0$ on ∂D.

(B) $\gamma(x') < 0$ on $M := \{x' \in \partial D : \mu(x') = 0\}$.

5.2 Proof of Estimate (1.2)

The proof of Estimate (1.2) is divided into three steps.

Step I: It suffices to show that estimate (1.2) holds true for some $\lambda_0 > 0$, since we have, for all $\lambda \in \mathbf{C}$,

$$(A - \lambda_0)u = (A - \lambda)u + (\lambda - \lambda_0)u.$$

We take a positive constant λ_0 so large that the function $c(x) - \lambda_0$ satisfies the condition

$$c(x) - \lambda_0 < 0 \quad \text{on the double } \widehat{D} \text{ of } D. \tag{5.1}$$

This condition (5.1) implies that condition (4.1) is satisfied for the operator $A - \lambda_0$. Therefore, by applying Theorems 4.4 and 4.3 to the operator $A - \lambda_0$ we can obtain the following two fundamental results:

(a) The Dirichlet problem

$$\begin{cases} (A - \lambda_0)w = 0 & \text{in } D, \\ w = \varphi & \text{on } \partial D \end{cases}$$

has a unique solution $w \in H^{t,p}(D)$ for any function $\varphi \in B^{t-1/p,p}(\partial D)$ with $t \in \mathbf{R}$.

(b) The Poisson operator

$$P(\lambda_0) : B^{t-1/p,p}(\partial D) \longrightarrow H^{t,p}(D),$$

defined by $w = P(\lambda_0)\varphi$, is an isomorphism of the space $B^{t-1/p,p}(\partial D)$ onto the null space $\mathcal{N}(A - \lambda_0, t, p) = \{u \in H^{t,p}(D) : (A - \lambda_0)u = 0 \text{ in } D\}$ for all $t \in \mathbf{R}$; and its inverse is the trace operator γ_0 on the boundary ∂D.

We let

$$\begin{aligned} T(\lambda_0) : \ C^\infty(\partial D) &\longrightarrow C^\infty(\partial D) \\ \varphi &\longmapsto LP(\lambda_0)\varphi. \end{aligned}$$

Then we have the formula

$$T(\lambda_0) = \mu(x')\Pi(\lambda_0) + \gamma(x'),$$

where

$$\Pi(\lambda_0)\varphi = \left. \frac{\partial}{\partial \mathbf{n}} (P(\lambda_0)\varphi) \right|_{\partial D}.$$

It is known that the operator $\Pi(\lambda_0)$ is a classical pseudo-differential operator of first order on the boundary ∂D and that its complete symbol is given by the following formula (cf. [Ta2, Section 10.2]):

$$\begin{aligned} &\left(p_1(x', \xi') + \sqrt{-1}\, q_1(x', \xi')\right) + \left(p_0(x', \xi') + \sqrt{-1}\, q_0(x', \xi')\right) \\ &+ \text{terms of order} \leq -1 \text{ depending on } \lambda_0, \end{aligned}$$

where

$$p_1(x',\xi') < 0 \quad \text{on the bundle } T^*(\partial D)\setminus\{0\} \text{ of non-zero cotangent vectors.} \tag{5.2}$$

For example, if A is the usual Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_N^2},$$

then we have the formula

$$p_1(x',\xi') = \text{minus the length } |\xi'| \text{ of } \xi' \text{ with respect to the Riemannian metric of } \partial D \text{ induced by the natural metric of } \mathbf{R}^N.$$

Therefore, we obtain that the operator $T(\lambda_0) = \mu(x')\Pi(\lambda_0) + \gamma(x')$ is a classical pseudo-differential operator of first order on the boundary ∂D and further that its complete symbol $t(x',\xi';\lambda_0)$ is given by the following formula:

$$\begin{aligned} t(x',\xi';\lambda_0) = {} & \mu(x')\left(p_1(x',\xi') + \sqrt{-1}\, q_1(x',\xi')\right) \\ & + \left([\gamma(x') + \mu(x')p_0(x',\xi')] + \sqrt{-1}\,\mu(x')q_0(x',\xi')\right) \\ & + \text{terms of order} \leq -1 \text{ depending on } \lambda_0. \end{aligned} \tag{5.3}$$

Then, by arguing just as in Section 4.3 we can prove that the question of the validity of *a priori* estimates and the question of regularity for solutions of problem (1.1) for $\lambda = \lambda_0$ are reduced to the corresponding questions for the operator $T(\lambda_0)$ (cf. Theorems 4.9 and 4.10).

Step II: Therefore, in order to prove estimate (1.2) for $\lambda = \lambda_0$ it suffices to show the following:

Lemma 5.1. *Assume that conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ *on* ∂D.
(B) $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then we have, for all $s \in \mathbf{R}$,

$$\varphi \in \mathcal{D}'(\partial D),\ T(\lambda_0)\varphi \in B^{s,p}(\partial D) \implies \varphi \in B^{s,p}(\partial D). \tag{5.4}$$

Furthermore, for any $t < s$, *there exists a positive constant* $C_{s,t}$ *such that*

$$|\varphi|_{s,p} \leq C_{s,t}\left(|T(\lambda_0)\varphi|_{s,p} + |\varphi|_{t,p}\right). \tag{5.5}$$

Proof. (a) The proof of Lemma 5.1 is based on the following lemma (cf. [Ka, Theorem 3.1]):

Lemma 5.2. *Assume that conditions (A) and (B) are satisfied. Then, for each point x' of ∂D, we can find a neighborhood $U(x')$ of x' such that:*

For any compact $K \subset U(x')$ and any multi-indices α, β, there exist positive constants $C_{K,\alpha,\beta}$ and C_K such that we have, for all $x' \in K$ and all $|\xi'| \geq C_K$,

$$\left| D_{\xi'}^{\alpha} D_{x'}^{\beta} t(x', \xi'; \lambda_0) \right| \leq C_{K,\alpha,\beta} \left| t(x', \xi'; \lambda_0) \right| (1 + |\xi|)^{-|\alpha| + (1/2)|\beta|}, \tag{5.6a}$$

$$\left| t(x', \xi'; \lambda_0) \right|^{-1} \leq C_K. \tag{5.6b}$$

Granting Lemma 5.2 for the moment, we shall prove Lemma 5.1.

(b) First, we cover ∂D by a finite number of local charts $\{(U_j, \chi_j)\}_{j=1}^{m}$ in each of which inequalities (5.6a) and (5.6b) hold true. Since the operator $T(\lambda_0)$ satisfies conditions (3.7a) and (3.7b) of Theorem 3.16 with $\mu := 0$, $\rho := 1$ and $\delta = 1/2$, it follows from an application of the same theorem that there exists a *parametrix* $S(\lambda_0)$ in the class $L^0_{1,1/2}(U_j)$ for $T(\lambda_0)$:

$$\begin{cases} T(\lambda_0) S(\lambda_0) \equiv I \mod L^{-\infty}(U_j), \\ S(\lambda_0) T(\lambda_0) \equiv I \mod L^{-\infty}(U_j). \end{cases}$$

Let $\{\varphi_j\}_{j=1}^{m}$ be a partition of unity subordinate to the covering $\{U_j\}_{j=1}^{m}$, and choose a function $\psi_j(x') \in C_0^{\infty}(U_j)$ such that $\psi_j(x') = 1$ on $\operatorname{supp} \varphi_j$, so that $\varphi_j(x') \psi_j(x') = \varphi_j(x')$.

Now we may assume that $\varphi \in B^{t,p}(\partial D)$ for some $t < s$ and that $T(\lambda_0)\varphi \in B^{s,p}(\partial D)$. We remark that the operator $T(\lambda_0)$ can be written in the following form:

$$T(\lambda_0) = \sum_{j=1}^{m} \varphi_j T(\lambda_0) \psi_j + \sum_{j=1}^{m} \varphi_j T(\lambda_0)(1 - \psi_j).$$

However, by applying Theorems 3.12 and 3.8 to our situation we obtain that the second terms $\varphi_j T(\lambda_0)(1 - \psi_j)$ are in $L^{-\infty}(\partial D)$. Indeed, it suffices to note that

$$\varphi_j(x') \left(1 - \psi_j(x')\right) = \varphi_j(x') - \varphi_j(x') = 0.$$

Hence we are reduced to the study of the first terms $\varphi_j T(\lambda_0) \psi_j$. This implies that we have only to prove the following *local* version of assertions (5.4) and (5.5):

$$\psi_j \varphi \in B^{t,p}(U_j),\, T(\lambda_0)\psi_j \varphi \in B^{s,p}(U_j) \implies \psi_j \varphi \in B^{s,p}(U_j). \tag{5.7}$$

$$\left| \psi_j \varphi \right|_{s,p} \leq C'_{s,t} \left(\left| T(\lambda_0) \psi_j \varphi \right|_{s,p}^{2} + \left| \psi_j \varphi \right|_{t,p}^{2} \right). \tag{5.8}$$

However, by applying the Besov-space boundedness theorem (Theorem 3.15) to our situation we obtain that the parametrix $S(\lambda_0)$ maps $B^{\sigma,p}_{\mathrm{loc}}(U_j)$ continuously into itself for all $\sigma \in \mathbf{R}$. This proves the desired assertions (5.7) and (5.8), since we have the assertion

$$\psi_j \varphi \equiv S(\lambda_0) \left(T(\lambda_0) \psi_j \varphi \right) \mod C^{-\infty}(U_j).$$

Lemma 5.1 is proved, apart from the proof of Lemma 5.2.

Step III: **Proof of Lemma 5.2**

The proof of Lemma 5.2 is divided into five steps.

Step III-1: First, we verify condition (5.6b).

By assertions (5.3) and (5.2), we can find positive constants c_0 and c_1 such that we have, for all sufficiently large $|\xi'|$,

$$
\begin{aligned}
|t(x', \xi'; \lambda_0)| &\geq \mu(x')\, |p_1(x', \xi') + p_0(x', \xi')| - \gamma(x') \\
&\geq \begin{cases} c_0\, \mu(x')|\xi'| - \frac{1}{2}\gamma(x') & \text{if } 0 \leq \mu(x') \leq c_1, \\ \frac{c_0}{2}\, \mu(x')|\xi'| - \gamma(x') & \text{if } c_1 \leq \mu(x') \leq 1, \end{cases}
\end{aligned}
$$

since $\gamma(x') < 0$ on $M = \{x' \in \partial D : \mu(x') = 0\}$. Hence we have, for all sufficiently large $|\xi'|$,

$$
|t(x', \xi'; \lambda_0)| \geq C\, (\mu(x')|\xi'| + 1)\,. \tag{5.9}
$$

Here in the following the letter C denotes a generic positive constant.

Inequality (5.9) implies the desired condition (5.6b):

$$
|t(x', \xi'; \lambda_0)| \geq C. \tag{5.10}
$$

Step III-2: Next we verify condition (5.6a) for $|\alpha| = 1$ and $|\beta| = 0$.

Since we have, for all sufficiently large $|\xi'|$,

$$
\left|D_{\xi'}^{\alpha} t(x', \xi'; \lambda_0)\right| \leq C\left(\mu(x') + |\xi'|^{-1}\right),
$$

it follows from inequality (5.9) that

$$
\begin{aligned}
\left|D_{\xi'}^{\alpha} t(x', \xi'; \lambda_0)\right| &\leq C(1 + |\xi'|)^{-1}\, (\mu(x')|\xi'| + 1) \\
&\leq C(1 + |\xi'|)^{-1}\, |t(x', \xi'; \lambda_0)|\,.
\end{aligned}
$$

This inequality proves the desired condition (5.6a) for $|\alpha| = 1$ and $|\beta| = 0$.

Step III-3: We verify condition (5.6a) for $|\beta| = 1$ and $|\alpha| = 0$. To do this, we need the following elementary lemma on non-negative functions.

Lemma 5.3. *Let $f(x)$ be a non-negative, C^2 function on $\mathbf{R}$ such that we have, for some positive constant c,*

$$
\sup_{x \in \mathbf{R}} |f''(x)| \leq c. \tag{5.11}
$$

Then we have the inequality

$$
|f'(x)| \leq \sqrt{2c}\,\sqrt{f(x)} \quad \text{on } \mathbf{R}. \tag{5.12}
$$

Proof. In view of Taylor's formula, it follows that

$$0 \leq f(y) = f(x) + f'(x)(y-x) + \frac{f''(\xi)}{2}(y-x)^2,$$

where ξ is between x and y. Thus, by letting $z = x - y$ we obtain from estimate (5.11) that

$$\begin{aligned} 0 &\leq f(x) + f'(x)z + \frac{f''(\xi)}{2} z^2 \\ &\leq f(x) + f'(x)z + \frac{c}{2} z^2 \quad \text{for all } z \in \mathbf{R}. \end{aligned}$$

This implies the desired inequality (5.12) if we take the discriminant of the quadratic equation. □

Step III-4: Since we have, for all sufficiently large $|\xi'|$,

$$\left| D_{x'}^{\beta} t(x', \xi'; \lambda_0) \right| \leq C \left(\left| D_{x'}^{\beta} \mu(x') \right| \cdot |\xi'| + \mu(x')|\xi'| + 1 \right),$$

it follows from an application of Lemma 5.3 and inequalities (5.9) and (5.10) that

$$\begin{aligned} \left| D_{x'}^{\beta} t(x', \xi'; \lambda_0) \right| &\leq C \left[\left(\sqrt{\mu(x')}\, |\xi'| + 1 \right) + (\mu(x')|\xi'| + 1) \right] \\ &\leq C \left[|\xi'|^{1/2} \left(\mu(x')|\xi'| + 1 \right)^{1/2} + (\mu(x')|\xi'| + 1) \right] \\ &\leq C \left| t(x', \xi'; \lambda_0) \right| \left(|\xi'|^{1/2} \left| t(x', \xi'; \lambda_0) \right|^{-1/2} + 1 \right) \\ &\leq C \left| t(x', \xi'; \lambda_0) \right| \left(1 + |\xi'| \right)^{1/2}. \end{aligned}$$

This inequality proves the desired condition (5.6a) for $|\beta| = 1$ and $|\alpha| = 0$.

Step III-5: Similarly, we can verify condition (5.6a) for the general case: $|\alpha| + |\beta| = k$, $k \in \mathbf{N}$.

Now the proof of Lemma 5.1 and hence that of Theorem 1.1 is complete. □

6

A Priori Estimates

This Chapter 6 and the next Chapter 7 are devoted to the proof of Theorem 1.2. In this chapter we study the operator A_p, and prove *a priori* estimates for the operator $A_p - \lambda I$ (Theorem 6.3) which will play a fundamental role in the next chapter. In the proof we make good use of Agmon's method (Proposition 6.4). This is a technique of treating a spectral parameter λ as a second-order, elliptic differential operator of an extra variable and relating the old problem to a new problem with the additional variable.

Recall that the operator A_p is a unbounded linear operator from $L^p(D)$ into itself defined by the following formulas:

(a) The domain of definition $\mathcal{D}(A_p)$ of A_p is the space

$$\mathcal{D}(A_p) = \left\{ u \in H^{2,p}(D) = W^{2,p}(D) : Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \right\}. \tag{1.3}$$

(b) $A_p u = Au$, $u \in \mathcal{D}(A_p)$.

We remark that the operator A_p is densely defined, since the domain $\mathcal{D}(A_p)$ contains the space $C_0^\infty(D)$.

First, we have the following:

Lemma 6.1. *Assume that conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ *on* ∂D.
(B) $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then we have the a priori *estimate*

$$\|u\|_{2,p} \leq C \left(\|Au\|_p + \|u\|_p \right), \quad u \in \mathcal{D}(A_p). \tag{6.1}$$

Proof. The *a priori* estimate (6.1) follows immediately from estimate (1.2) of Theorem 1.1 with $\varphi := 0$. □

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Corollary 6.2. *The operator A_p is a closed operator.*

Proof. Let $\{u_j\}$ be an arbitrary sequence in the domain $\mathcal{D}(A_p)$ such that:

$$\begin{cases} u_j \longrightarrow u & \text{in } L^p(D), \\ Au_j \longrightarrow v & \text{in } L^p(D). \end{cases}$$

Then, by applying estimate (6.1) to the sequence $\{u_j\}$ we find that $\{u_j\}$ is a Cauchy sequence in the space $W^{2,p}(D)$. This proves that

$$u \in W^{2,p}(D),$$

and that

$$u_j \longrightarrow u \quad \text{in } W^{2,p}(D).$$

Hence we have the formula

$$Au = \lim_{j\to\infty} Au_j = v \quad \text{in } L^p(D),$$

and also, by Proposition 4.7 with $B := L$,

$$Lu = \lim_{j\to\infty} Lu_j = 0 \quad \text{in } B^{1-1/p,p}(\partial D).$$

Summing up, we have proved that $u \in \mathcal{D}(A_p)$ and $A_p u = v$.

The proof of Corollary 6.2 is complete. □

The next theorem is an essential step in the proof of Theorem 1.2 (cf. the proof of Theorem 7.1 in Chapter 7):

Theorem 6.3. *Assume that conditions (A) and (B) are satisfied. Then, for every $-\pi < \theta < \pi$, there exists a positive constant $R(\theta)$ depending on θ such that if $\lambda = r^2 e^{i\theta}$ and $|\lambda| = r^2 \geq R(\theta)$, we have, for all $u \in W^{2,p}(D)$ satisfying $Lu = 0$ on ∂D (i. e., $u \in \mathcal{D}(A_p)$),*

$$|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq C(\theta) \, \|(A-\lambda)u\|_p, \tag{6.2}$$

with a positive constant $C(\theta)$ depending on θ. Here $|\cdot|_{j,p}$, $j = 1, 2$, is the seminorm on the space $W^{2,p}(D)$ defined by the formula

$$|u|_{j,p} = \left(\int_D \sum_{|\alpha|=j} |D^\alpha u(x)|^p dx \right)^{1/p}.$$

Proof. The proof of Theorem 6.3 is divided into two steps.

Step I: We shall make use of a method essentially due to Agmon (cf. [Ag], [Fu], [LM], [Ta1]).

We introduce an auxiliary variable y of the unit circle

$$S = \mathbf{R}/2\pi\mathbf{Z},$$

and replace the complex parameter λ by the second-order differential operator

$$-e^{i\theta}\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi.$$

Namely, we replace the operator $A - \lambda$ by the operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta}\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi,$$

and consider instead of the problem with spectral parameter

$$\begin{cases} (A-\lambda)u = f & \text{in } D, \\ Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 & \text{on } \partial D \end{cases} \tag{1.1}$$

the following boundary value problem:

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \left(A + e^{i\theta}\frac{\partial^2}{\partial y^2}\right)\widetilde{u} = \widetilde{f} & \text{in } D \times S, \\ L\widetilde{u} = \mu(x')\frac{\partial \widetilde{u}}{\partial \mathbf{n}} + \gamma(x')\widetilde{u} = 0 & \text{on } \partial D \times S. \end{cases} \tag{6.3}$$

We remark that the operator $\widetilde{\Lambda}(\theta)$ is *elliptic* for $-\pi < \theta < \pi$.

Then we have the following result, analogous to Lemma 6.1:

Proposition 6.4. *Assume that conditions (A) and (B) are satisfied. Then we have, for all $\widetilde{u} \in W^{2,p}(D \times S)$ satisfying $L\widetilde{u} = 0$ on $\partial D \times S$,*

$$\|\widetilde{u}\|_{2,p} \leq \widetilde{C}(\theta)\left(\left\|\widetilde{\Lambda}(\theta)\widetilde{u}\right\|_p + \|\widetilde{u}\|_p\right), \tag{6.4}$$

with a positive constant $\widetilde{C}(\theta)$ depending on θ.

Proof. We reduce the study of problem (6.3) to that of a pseudo-differential operator on the boundary, just as in problem (1.1).

We can prove that Theorems 4.3 and 4.4 remain valid for the operator $\widetilde{\Lambda}(\theta) = A + e^{i\theta}\partial^2/\partial y^2$, $-\pi < \theta < \pi$:

($\widetilde{a}$) The Dirichlet problem

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{w} = 0 & \text{in } D \times S, \\ \widetilde{w} = \widetilde{\varphi} & \text{on } \partial D \times S \end{cases}$$

has a unique solution $\widetilde{w} \in H^{t,p}(D \times S)$ for any function $\widetilde{\varphi} \in B^{t-1/p,p}(\partial D \times S)$ with $t \in \mathbf{R}$.

($\widetilde{b}$) The Poisson operator

$$\widetilde{P}(\theta) : B^{t-1/p,p}(\partial D \times S) \longrightarrow H^{t,p}(D \times S),$$

defined by $\widetilde{w} = \widetilde{P}(\theta)\widetilde{\varphi}$, is an isomorphism of the space $B^{t-1/p,p}(\partial D \times S)$ onto the null space $\mathcal{N}(\widetilde{\Lambda}(\theta), t, p) = \{\widetilde{u} \in H^{t,p}(D \times S) : \widetilde{\Lambda}(\theta)\widetilde{u} = 0 \text{ in } D \times S\}$ for all $t \in \mathbf{R}$; and its inverse is the trace operator on the boundary $\partial D \times S$.

We let

$$\widetilde{T}(\theta): \ C^\infty(\partial D \times S) \longrightarrow C^\infty(\partial D \times S)$$
$$\widetilde{\varphi} \longmapsto L\widetilde{P}(\theta)\widetilde{\varphi}.$$

Then the operator $\widetilde{T}(\theta)$ can be decomposed as follows:

$$\widetilde{T}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + \gamma(x'), \tag{6.5}$$

where

$$\widetilde{\Pi}(\theta)\widetilde{\varphi} = \frac{\partial}{\partial \mathbf{n}}\left(\widetilde{P}(\theta)\widetilde{\varphi}\right)\Big|_{\partial D\times S}, \quad \widetilde{\varphi} \in C^\infty(\partial D \times S).$$

It is known that the operator $\widetilde{\Pi}(\theta)$ is a classical pseudo-differential operator of first order on the boundary $\partial D \times S$ and that its complete symbol is given by the following formula (cf. [Ta2, Section 10.2]):

$$\begin{aligned} &\left(\widetilde{p}_1(x',\xi',y,\eta;\theta) + \sqrt{-1}\ \widetilde{q}_1(x',\xi',y,\eta;\theta)\right) \\ &+ \left(\widetilde{p}_0(x',\xi',y,\eta;\theta) + \sqrt{-1}\ \widetilde{q}_0(x',\xi',y,\eta;\theta)\right) + \text{ terms of order} \leq -1, \end{aligned}$$

where

$$\begin{aligned} &\widetilde{p}_1(x',\xi',y,\eta;\theta) < 0 \\ &\text{on the bundle } T^*(\partial D \times S) \setminus \{0\} \text{ of non-zero cotangent vectors,} \\ &\text{for } -\pi < \theta < \pi. \end{aligned} \tag{6.6}$$

For example, if A is the usual Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_N^2},$$

then we have the formula

$$\begin{aligned} &\widetilde{p}_1(x',\xi',y,\eta;\theta) \\ &= -\left[\frac{\left[\left(|\xi'|^2 + \cos\theta\cdot\eta^2\right)^2 + \sin^2\theta\cdot\eta^4\right]^{1/2} + \left(|\xi'|^2 + \cos\theta\cdot\eta^2\right)}{2}\right]^{1/2}. \end{aligned}$$

Therefore, we obtain that the operator $\widetilde{T}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + \gamma(x')$ is a classical pseudo-differential operator of first order on the boundary $\partial D \times S$ and that its complete symbol is given by the following formula:

$$\begin{aligned} &\mu(x')\left(\widetilde{p}_1(x',\xi',y,\eta;\theta) + \sqrt{-1}\ \widetilde{q}_1(x',\xi',y,\eta;\theta)\right) \\ &+ \left([\gamma(x') + \mu(x')\widetilde{p}_0(x',\xi',y,\eta;\theta)] + \sqrt{-1}\,\mu(x')\widetilde{q}_0(x',\xi',y,\eta;\theta)\right) \\ &+ \text{terms of order} \leq -1. \end{aligned} \tag{6.7}$$

Then, by virtue of assertions (6.7) and (6.6) we can verify that the operator $\widetilde{T}(\theta)$ satisfies conditions (3.7a) and (3.7b) of Theorem 3.16 with $\mu := 0$, $\rho := 1$ and $\delta := 1/2$, just as in the proof of Lemma 5.2. Hence we obtain the following result, analogous to Lemma 5.1:

Lemma 6.5. *Assume that conditions (A) and (B) are satisfied. Then we have, for all* $s \in \mathbf{R}$,

$$\widetilde{\varphi} \in \mathcal{D}'(\partial D \times S),\ \widetilde{T}(\theta)\widetilde{\varphi} \in B^{s,p}(\partial D \times S) \implies \widetilde{\varphi} \in B^{s,p}(\partial D \times S).$$

Furthermore, for any $t < s$, *there exists a positive constant* $\widetilde{C}_{s,t}$ *such that*

$$|\widetilde{\varphi}|_{s,p} \leq \widetilde{C}_{s,t}\left(|\widetilde{T}(\theta)\widetilde{\varphi}|_{s,p} + |\widetilde{\varphi}|_{t,p}\right). \tag{6.8}$$

The desired estimate (6.4) follows from estimate (6.8) with $s := 2 - 1/p$ and $t := -1/p$, just as in the proof of Theorem 4.10.

The proof of Proposition 6.4 is complete. □

Step II: Now let $u(x)$ be an arbitrary function in the domain $\mathcal{D}(A_p)$:

$$u \in W^{2,p}(D) \text{ and } Lu = 0 \text{ on } \partial D.$$

We choose a function $\zeta(y)$ in $C^\infty(S)$ such that

$$\begin{cases} 0 \leq \zeta(y) \leq 1 \quad \text{on } S, \\ \operatorname{supp}\zeta \subset \left[\frac{\pi}{3}, \frac{5\pi}{3}\right], \\ \zeta(y) = 1 \quad \text{for} \quad \frac{\pi}{2} \leq y \leq \frac{3\pi}{2}, \end{cases}$$

and let

$$\widetilde{v}_\eta(x,y) = u(x) \otimes \zeta(y)e^{i\eta y}, \quad x \in D,\ y \in S,\ \eta \geq 0.$$

Then we have the assertions

$$\begin{aligned} \widetilde{v}_\eta &\in W^{2,p}(D \times S), \\ \widetilde{\Lambda}(\theta)\widetilde{v}_\eta &= \left(A + e^{i\theta}\frac{\partial^2}{\partial y^2}\right)\widetilde{v}_\eta \\ &= (A - \eta^2 e^{i\theta})u \otimes \zeta(y)e^{i\eta y} + 2(i\eta)e^{i\theta}u \otimes \zeta'(y)e^{i\eta y} + e^{i\theta}u \otimes \zeta''(y)e^{i\eta y}, \end{aligned}$$

and also

$$L\widetilde{v}_\eta(x',y) = Lu(x') \otimes \zeta(y)e^{i\eta y} = 0 \quad \text{on } \partial D \times S.$$

Thus, by applying inequality (6.4) to the functions

$$\widetilde{v}_\eta(x,y) = u(x) \otimes \zeta(y)e^{i\eta y}, \quad x \in D,\ y \in S,\ \eta \geq 0,$$

we obtain that

$$\left\|u \otimes \zeta e^{i\eta y}\right\|_{2,p} \leq \widetilde{C}(\theta)\left(\left\|\widetilde{\Lambda}(\theta)(u \otimes \zeta e^{i\eta y})\right\|_p + \left\|u \otimes \zeta e^{i\eta y}\right\|_p\right). \tag{6.9}$$

We can estimate each term of inequality (6.9) as follows:

$$\left\|u \otimes \zeta e^{i\eta y}\right\|_p = \left(\int_{D\times S} |u(x)|^p |\zeta(y)|^p \, dxdy\right)^{1/p} = \|\zeta\|_p \cdot \|u\|_p. \quad (6.10)$$

$$\begin{aligned}\left\|\widetilde{\Lambda}(\theta)(u \otimes \zeta e^{i\eta y})\right\|_p &\le \left\|(A - \eta^2 e^{i\theta})u \otimes \zeta e^{i\eta y}\right\|_p \\ &\quad + 2\eta \left\|u \otimes \zeta' e^{i\eta y}\right\|_p + \left\|u \otimes \zeta'' e^{i\eta y}\right\|_p \\ &\le \|\zeta\|_p \cdot \left\|(A - \eta^2 e^{i\theta})u\right\|_p \\ &\quad + (2\eta\|\zeta'\|_p + \|\zeta''\|_p)\,\|u\|_p. \end{aligned} \quad (6.11)$$

$$\begin{aligned}\left\|u \otimes \zeta e^{i\eta y}\right\|_{2,p}^p &= \sum_{|\alpha|\le 2} \int_{D\times S} \left|D_{x,y}^\alpha (u(x) \otimes \zeta(y) e^{i\eta y})\right|^p dxdy \\ &\ge \sum_{|\alpha|\le 2} \int_D \int_{\pi/2}^{3\pi/2} \left|D_{x,y}^\alpha (u(x) \otimes e^{i\eta y})\right|^p dxdy \\ &= \sum_{k+|\beta|\le 2} \int_D \int_{\pi/2}^{3\pi/2} \left|\eta^k D^\beta u(x)\right|^p dxdy \\ &\ge \pi \Bigg(\sum_{|\beta|=2} \int_D \left|D^\beta u(x)\right|^p dx + \eta^p \sum_{|\beta|=1} \int_D \left|D^\beta u(x)\right|^p dx \\ &\quad + \eta^{2p} \int_D |u(x)|^p dx \Bigg) \\ &= \pi \left(|u|_{2,p}^p + \eta^p |u|_{1,p}^p + \eta^{2p} \|u\|_p^p \right). \end{aligned} \quad (6.12)$$

Therefore, by carrying these three inequalities (6.10), (6.11) and (6.12) into inequality (6.9) we obtain that, with a positive constant $\widetilde{C}'(\theta)$ independent of η,

$$|u|_{2,p} + \eta\, |u|_{1,p} + \eta^2 \|u\|_p \le \widetilde{C}'(\theta) \left(\left\|(A - \eta^2 e^{i\theta})u\right\|_p + \eta\, \|u\|_p \right).$$

If η is so large that

$$\eta \ge 2\widetilde{C}'(\theta),$$

then we can eliminate the last term on the right-hand side to obtain that

$$|u|_{2,p} + \eta\, |u|_{1,p} + \eta^2 \|u\|_p \le 2\widetilde{C}'(\theta) \left\|(A - \eta^2 e^{i\theta})u\right\|_p.$$

This proves the desired inequality (6.2) if we take

$$\begin{aligned} \lambda &:= \eta^2 e^{i\theta}, \\ R(\theta) &:= 4\widetilde{C}'(\theta)^2, \\ C(\theta) &:= 2\widetilde{C}'(\theta). \end{aligned}$$

The proof of Theorem 6.3 is now complete. □

7

Proof of Theorem 1.2

In this chapter we prove Theorem 1.2 (Theorems 7.1 and 7.9). Once again we make use of Agmon's method in the proof of Theorems 7.1 and 7.9. In particular, Agmon's method plays an important role in the proof of the *surjectivity* of the operator $A_p - \lambda I$ (Proposition 7.2).

7.1 Proof of Theorem 1.2, Part (i)

First, we prove part (i) of Theorem 1.2:

Theorem 7.1. *Assume that conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ *on* ∂D.
(B) $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then, for every $0 < \varepsilon < \pi/2$, *there exists a positive constant* $r_p(\varepsilon)$ *such that the resolvent set of* A_p *contains the set*

$$\Sigma_p(\varepsilon) = \{\lambda = r^2 e^{i\theta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\},$$

and that the resolvent $(A_p - \lambda I)^{-1}$ *satisfies the estimate*

$$\left\|(A_p - \lambda I)^{-1}\right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon), \tag{1.4}$$

where $c_p(\varepsilon)$ *is a positive constant depending on* ε.

Proof. The proof of Theorem 7.1 is divided into three steps.

Step I: By estimate (6.2), it follows that if $\lambda = r^2 e^{i\theta}$, $-\pi < \theta < \pi$ and if $|\lambda| = r^2 \geq R(\theta)$, then we have, for all $u \in \mathcal{D}(A_p)$,

$$|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq C(\theta) \|(A_p - \lambda I)u\|_p.$$

However, we find from the proof of Theorem 6.3 that the constants $R(\theta)$ and $C(\theta)$ depend *continuously* on $\theta \in (-\pi, \pi)$, so that they may be chosen

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uniformly in $\theta \in [-\pi+\varepsilon, \pi-\varepsilon]$, for every $\varepsilon > 0$. This proves the existence of the constants $r_p(\varepsilon)$ and $c_p(\varepsilon)$, that is, we have, for all $\lambda = r^2 e^{i\theta}$ satisfying $r \geq r_p(\varepsilon)$ and $-\pi+\varepsilon \leq \theta \leq \pi-\varepsilon$,

$$|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq c_p(\varepsilon)\|(A_p - \lambda I)u\|_p. \tag{7.1}$$

By estimate (7.1), it follows that the operator $A_p - \lambda I$ is injective and its range $\mathcal{R}(A_p - \lambda I)$ is closed in $L^p(D)$, for all $\lambda \in \Sigma_p(\varepsilon)$.

Step II: We show that the operator $A_p - \lambda I$ is surjective for all $\lambda \in \Sigma_p(\varepsilon)$, that is,

$$\mathcal{R}(A_p - \lambda I) = L^p(D) \quad \text{for all } \lambda \in \Sigma_p(\varepsilon). \tag{7.2}$$

To do this, it suffices to show that the operator $A_p - \lambda I$ is a Fredholm operator with

$$\operatorname{ind}(A_p - \lambda I) = 0 \quad \text{for all } \lambda \in \Sigma_p(\varepsilon), \tag{7.3}$$

since $A_p - \lambda I$ is injective for all $\lambda \in \Sigma_p(\varepsilon)$.

Here we recall that a densely defined, closed linear operator T with domain $\mathcal{D}(T)$ from a Banach space X into itself is called a *Fredholm operator* if it satisfies the following three conditions:

(i) The null space $\mathcal{N}(T) = \{x \in \mathcal{D}(T) : Tx = 0\}$ of T has finite dimension, that is, $\dim \mathcal{N}(T) < \infty$.
(ii) The range $\mathcal{R}(T) = \{Tx : x \in \mathcal{D}(T)\}$ of T is closed in X.
(iii) The range $\mathcal{R}(T)$ has finite codimension in X, that is, $\operatorname{codim} \mathcal{R}(T) = \dim X/\mathcal{R}(T) < \infty$.

In this case the *index* $\operatorname{ind} T$ of T is defined by the formula

$$\operatorname{ind} T = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T).$$

Step II-1: We reduce the study of the operator $A_p - \lambda I$ $(\lambda \in \Sigma_p(\varepsilon))$ to that of a pseudo-differential operator on the boundary, just as in the proof of Theorem 1.1.

Let $T(\lambda)$ be a classical pseudo-differential operator of first order on the boundary ∂D defined as follows:

$$T(\lambda) = LP(\lambda) = \mu(x')\Pi(\lambda) + \gamma(x'), \quad \lambda \in \Sigma_p(\varepsilon), \tag{7.4}$$

where

$$\begin{aligned} \Pi(\lambda) : \quad C^\infty(\partial D) &\longrightarrow C^\infty(\partial D) \\ \varphi &\longmapsto \frac{\partial}{\partial \mathbf{n}}(P(\lambda)\varphi)\Big|_{\partial D}. \end{aligned}$$

Since the operator $T(\lambda) : C^\infty(\partial D) \to C^\infty(\partial D)$ extends to a continuous linear operator $T(\lambda) : B^{t,p}(\partial D) \to B^{t-1,p}(\partial D)$ for all $t \in \mathbf{R}$, we can introduce a densely defined, closed linear operator

$$\mathcal{T}_p(\lambda) : B^{2-1/p,p}(\partial D) \longrightarrow B^{2-1/p,p}(\partial D)$$

as follows.

(α) The domain $\mathcal{D}(\mathcal{T}_p(\lambda))$ of $\mathcal{T}_p(\lambda)$ is the space

$$\mathcal{D}(\mathcal{T}_p(\lambda)) = \left\{\varphi \in B^{2-1/p,p}(\partial D) : T(\lambda)\varphi \in B^{2-1/p,p}(\partial D)\right\}.$$

(β) $\mathcal{T}_p(\lambda)\varphi = T(\lambda)\varphi$, $\varphi \in \mathcal{D}(\mathcal{T}_p(\lambda))$.

Then we can obtain the following three results (cf. [Ta2, Section 8.3]):

(I) The null space $\mathcal{N}(A_p - \lambda I)$ of $A_p - \lambda I$ has finite dimension if and only if the null space $\mathcal{N}(\mathcal{T}_p(\lambda))$ of $\mathcal{T}_p(\lambda)$ has finite dimension, and we have the formula

$$\dim \mathcal{N}(A_p - \lambda I) = \dim \mathcal{N}(\mathcal{T}_p(\lambda)).$$

(II) The range $\mathcal{R}(A_p - \lambda I)$ of $A_p - \lambda I$ is closed if and only if the range $\mathcal{R}(\mathcal{T}_p(\lambda))$ of $\mathcal{T}_p(\lambda)$ is closed; and $\mathcal{R}(A_p - \lambda I)$ has finite codimension if and only if $\mathcal{R}(\mathcal{T}_p(\lambda))$ has finite codimension, and we have the formula

$$\operatorname{codim} \mathcal{R}(A_p - \lambda I) = \operatorname{codim} \mathcal{R}(\mathcal{T}_p(\lambda)).$$

(III) The operator $A_p - \lambda I$ is a Fredholm operator if and only if the operator $\mathcal{T}_p(\lambda)$ is a Fredholm operator, and we have the formula

$$\operatorname{ind}(A_p - \lambda I) = \operatorname{ind} \mathcal{T}_p(\lambda).$$

Therefore, the desired assertion (7.3) is reduced to the following assertion:

$$\operatorname{ind} \mathcal{T}_p(\lambda) = 0 \quad \text{for all } \lambda \in \Sigma_p(\varepsilon). \tag{7.5}$$

Step II-2: To prove assertion (7.5), we shall make use of Agmon's method just as in Chapter 6.

Let $\widetilde{T}(\theta)$ be the classical pseudo-differential operator of first order on the boundary $\partial D \times S$ introduced in Chapter 6 (see formula (6.5)):

$$\widetilde{T}(\theta) = L\widetilde{P}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + \gamma(x'), \quad -\pi < \theta < \theta,$$

where

$$\begin{aligned} \widetilde{\Pi}(\theta) : \ & C^\infty(\partial D \times S) \longrightarrow C^\infty(\partial D \times S) \\ & \tilde{\varphi} \longmapsto \frac{\partial}{\partial \mathbf{n}}\left(\widetilde{P}(\theta)\tilde{\varphi}\right)\Big|_{\partial D \times S}. \end{aligned}$$

We define a densely defined, closed linear operator

$$\widetilde{\mathcal{T}}_p(\theta) : B^{2-1/p,p}(\partial D \times S) \longrightarrow B^{2-1/p,p}(\partial D \times S)$$

as follows.

($\tilde{\alpha}$) The domain $\mathcal{D}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ of $\widetilde{\mathcal{T}}_p(\theta)$ is the space

$$\mathcal{D}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \left\{\tilde{\varphi} \in B^{2-1/p,p}(\partial D \times S) : \widetilde{T}(\theta)\tilde{\varphi} \in B^{2-1/p,p}(\partial D \times S)\right\}.$$

($\tilde{\beta}$) $\widetilde{\mathcal{T}}_p(\theta)\tilde{\varphi} = \widetilde{T}(\theta)\tilde{\varphi}$, $\tilde{\varphi} \in \mathcal{D}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$.

Then the most fundamental relationship between the operators $\widetilde{\mathcal{T}}_p(\theta)$ and $\mathcal{T}_p(\lambda)$ is stated as follows:

Proposition 7.2. *If* ind $\widetilde{\mathcal{T}}_p(\theta)$ *is finite, then there exists a* finite subset K *of* $\mathbf{Z}$ *such that the operator* $\mathcal{T}_p(\lambda')$ *is bijective for all* $\lambda' = \ell^2 e^{i\theta}$ *satisfying* $\ell \in \mathbf{Z} \setminus K$.

Granting Proposition 7.2 for the moment, we shall prove Theorem 7.1.

Step III: **End of Proof of Theorem 7.1**

Step III-1: We show that if conditions (A) and (B) are satisfied, then we have the assertion

$$\operatorname{ind} \widetilde{\mathcal{T}}_p(\theta) = \dim \mathcal{N}(\widetilde{\mathcal{T}}_p(\theta)) - \operatorname{codim} \mathcal{R}(\widetilde{\mathcal{T}}_p(\theta)) < \infty. \tag{7.6}$$

To this end, we need a useful criterion for Fredholm operators (cf. [Ta2, Theorem 3.7.6]):

Lemma 7.3 (Peetre). *Let X, Y, Z be Banach spaces such that $X \subset Z$ is a* compact *injection, and let T be a closed linear operator with $\mathcal{D}(T)$ from X into Y with domain $\mathcal{D}(T)$. Then the following two conditions are equivalent:*

(i) The null space $\mathcal{N}(T)$ of T has finite dimension and the range $\mathcal{R}(T)$ of T is closed in Y.
(ii) There is a positive constant C such that

$$\|x\|_X \leq C\left(\|Tx\|_Y + \|x\|_Z\right) \quad \text{for all } x \in \mathcal{D}(T). \tag{7.7}$$

Proof. (i) $\Longrightarrow$ (ii): Since the null space $\mathcal{N}(T)$ has finite dimension, we can find a closed topological complement X_0 in X:

$$X = \mathcal{N}(T) \oplus X_0. \tag{7.8}$$

This gives that

$$\mathcal{D}(T) = \mathcal{N}(T) \oplus \left(\mathcal{D}(T) \cap X_0\right).$$

Namely, every element x of $\mathcal{D}(T)$ can be written in the form

$$x = x_0 + x_1, \quad x_0 \in \mathcal{D}(T) \cap X_0,\ x_1 \in \mathcal{N}(T).$$

Moreover, since the range $\mathcal{R}(T)$ is closed in Y, it follows from an application of the closed graph theorem [Yo, Chapter II, Section 6, Theorem 1] that there exists a positive constant C such that

$$\|x_0\|_X \leq C\|Tx_0\|_Y = \|Tx\|_Y. \tag{7.9}$$

Here and in the following the letter C denotes a generic positive constant independent of x.

On the other hand, it should be noticed that all norms on a finite dimensional linear space are equivalent. This gives that

$$\|x_1\|_X \leq C\|x_1\|_Z. \tag{7.10}$$

Moreover, since the injection $X \to Z$ is compact and hence is continuous, we obtain that

$$\|x_1\|_Z \leq \|x\|_Z + \|x_0\|_Z \leq \|x\|_Z + C\|x_0\|_X. \tag{7.11}$$

Thus we have, by inequalities (7.10) and (7.11),

$$\|x_1\|_X \leq C\left(\|x\|_Z + \|x_0\|_X\right). \tag{7.12}$$

Therefore, by combining inequalities (7.9) and (7.12) we obtain the desired inequality

$$\|x\|_X \leq \|x_0\|_X + \|x_1\|_X \leq C\left(\|Tx\|_Y + \|x\|_Z\right) \quad \text{for all } x \in \mathcal{D}(T), \tag{7.7}$$

since $Tx_0 = Tx$.

(ii) $\Longrightarrow$ (i): By inequality (7.7), it follows that

$$\|x\|_X \leq C\|x\|_Z \quad \text{for all } x \in \mathcal{N}(T). \tag{7.13}$$

However, the null space $\mathcal{N}(T)$ is closed in X, and so it is a Banach space. Since the injection $X \to Z$ is compact, we obtain from inequality (7.13) that the closed unit ball $\{x \in \mathcal{N}(T) : \|x\|_X \leq 1\}$ of the Banach space $\mathcal{N}(T)$ is compact. Hence it follows from an application of [Yo, Chapter III, Section 2, Corollary 2] that

$$\dim \mathcal{N}(T) < \infty.$$

Let X_0 be a closed topological complement of $\mathcal{N}(T)$ as in decomposition (7.8).

To prove the closedness of $\mathcal{R}(T)$, it suffices to show that

$$\|x\|_X \leq C\|Tx\|_Y \quad \text{for all } x \in \mathcal{D}(T) \cap X_0.$$

Assume, to the contrary, that:

For every $n \in \mathbf{N}$, there is an element x_n of $\mathcal{D}(T) \cap X_0$ such that

$$\|x_n\|_X > n\|Tx_n\|_Y.$$

If we let

$$x_n' = \frac{x_n}{\|x_n\|_X},$$

then we have the assertions

$$x_n' \in \mathcal{D}(T) \cap X_0, \quad \|x_n'\|_X = 1, \tag{7.14a}$$

$$\|Tx_n'\|_Y < \frac{1}{n}. \tag{7.14b}$$

Since the injection $X \to Z$ is compact, by passing to a subsequence we may assume that the sequence $\{x'_n\}$ is a Cauchy sequence in Z. Then, by combining inequalities (7.14b) and (7.7) we find that the sequence $\{x'_n\}$ is a Cauchy sequence in X, and hence it converges to some element x' of $X_0 \subset X$. Thus, by assertions (7.14a) and (7.14b) it follows that

$$\|x'\|_X = \lim_n \|x'_n\|_X = 1,$$

and further that

$$x' \in \mathcal{D}(T), \quad Tx' = 0,$$

since the operator T is closed.

Summing up, we have proved that

$$\begin{aligned} &x' \in \mathcal{N}(T), \\ &\|x'\|_X = 1. \end{aligned}$$

However, this is a contradiction. Indeed, we then have the assertion

$$x' \in \mathcal{N}(T) \cap X_0 = \{0\}.$$

The proof of Lemma 7.3 is complete. $\square$

By using estimate (6.8) with $s := 2 - 1/p$, we obtain that

$$|\tilde{\varphi}|_{2-1/p,p} \leq \widetilde{C}_t \left(|\widetilde{T}(\theta)\tilde{\varphi}|_{2-1/p,p} + |\tilde{\varphi}|_{t,p} \right), \quad \tilde{\varphi} \in \mathcal{D}(\widetilde{\mathcal{T}}_p(\theta)), \tag{7.15}$$

where $t < 2 - 1/p$. However, it follows from an application of the Rellich–Kondrachov theorem that the injection $B^{2-1/p,p}(\partial D \times S) \to B^{t,p}(\partial D \times S)$ is *compact* for $t < 2 - 1/p$. Thus, by applying Peetre's lemma (Lemma 7.3) with

$$\begin{aligned} &X = Y := B^{2-1/p,p}(\partial D \times S), \\ &Z := B^{t,p}(\partial D \times S), \\ &T := \widetilde{\mathcal{T}}_p(\theta), \end{aligned}$$

we obtain that the range $\mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ is closed in $B^{2-1/p,p}(\partial D \times S)$ and that

$$\dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) < \infty. \tag{7.16}$$

On the other hand, by formula (6.7) we find that the complete symbol of the adjoint $\widetilde{T}(\theta)^*$ is given by the following formula (cf. Theorem 3.11):

$$\begin{aligned} &\mu(x')\left(\tilde{p}_1(x',\xi',y,\eta;\theta) - \sqrt{-1}\,\tilde{q}_1(x',\xi',y,\eta;\theta)\right) \\ &+\left(\left[\gamma(x') + \mu(x')\tilde{p}_0(x',\xi',y,\eta;\theta) - \sum_{j=1}^{n-1} \partial_{x_j}\left(\mu(x')\cdot\partial_{\xi_j}\tilde{q}_1(x',\xi',y,\eta;\theta)\right)\right]\right. \\ &\left.-\sqrt{-1}\left[\mu(x')\tilde{q}_0(x',\xi',y,\eta;\theta) + \sum_{j=1}^{n-1} \partial_{x_j}\left(\mu(x')\cdot\partial_{\xi_j}\tilde{p}_1(x',\xi',y,\eta;\theta)\right)\right]\right) \\ &+ \text{terms of order} \leq -1. \end{aligned}$$

However, it follows from an application of Lemma 5.3 that

$$\partial_{x_j}\mu(x') = 0 \text{ on } M = \{x' \in \partial D : \mu(x') = 0\}.$$

Thus we can easily verify that the pseudo-differential operator $\widetilde{T}(\theta)^*$ satisfies conditions (3.7a) and (3.7b) of Theorem 3.16 with $\mu := 0$, $\rho := 1$ and $\delta := 1/2$. This implies that estimate (7.15) holds true for the adjoint operator $\widetilde{\mathcal{T}}_p(\theta)^*$ of $\widetilde{\mathcal{T}}_p(\theta)$:

$$|\tilde{\psi}|_{-2+1/p,p'} \leq \widetilde{C}_\tau \left(|\widetilde{T}(\theta)^*\tilde{\psi}|_{-2+1/p,p'} + |\tilde{\psi}|_{\tau,p'} \right), \quad \tilde{\psi} \in \mathcal{D}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right),$$

where $\tau < -2 + 1/p$ and $p' = p/(p-1)$ is the exponent conjugate to p. Hence we have, by the closed range theorem ([Yo, Chapter VII, Section 5, Theorem]) and Peetre's lemma (Lemma 7.3),

$$\operatorname{codim} \mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right) < \infty, \tag{7.17}$$

since the injection $B^{-2+1/p,p'}(\partial D \times S) \to B^{\tau,p'}(\partial D \times S)$ is compact for $\tau < -2 + 1/p$.

Therefore, the desired assertion (7.6) follows by combining assertions (7.16) and (7.17).

Step III-2: By assertion (7.6), we can apply Proposition 7.2 to obtain that the operator $\mathcal{T}_p(\ell^2 e^{i\theta}) : B^{2-1/p,p}(\partial D) \to B^{2-1/p,p}(\partial D)$ is bijective if $\ell \in \mathbf{Z} \setminus K$ for some *finite subset* K of $\mathbf{Z}$. In particular, we have the assertion

$$\operatorname{ind} \mathcal{T}_p(\lambda_0) = 0 \quad \text{for all } \lambda_0 = \ell^2 e^{i\theta} \text{ with } \ell \in \mathbf{Z} \setminus K. \tag{7.18}$$

However, in view of formulas (7.4) and (5.3) it follows that, for any given λ, $\lambda_0 \in \Sigma_p(\varepsilon)$, we can find a classical pseudo-differential operator $K(\lambda, \lambda_0)$ of order -1 on the boundary ∂D such that

$$T(\lambda) = T(\lambda_0) + K(\lambda, \lambda_0).$$

Furthermore, it follows from an application of the Rellich–Kondrachov theorem that the operator

$$K(\lambda, \lambda_0) : B^{2-1/p,p}(\partial D) \longrightarrow B^{2-1/p,p}(\partial D)$$

is *compact*. Hence we have the assertion

$$\operatorname{ind} \mathcal{T}_p(\lambda) = \operatorname{ind} \mathcal{T}_p(\lambda_0) \quad \text{for all } \lambda,\ \lambda_0 \in \Sigma_p(\varepsilon). \tag{7.19}$$

Therefore, the desired assertion (7.5) (and hence assertion (7.3)) follows by combining assertions (7.18) and (7.19).

Step III-3: Summing up, we have proved that the operator $A_p - \lambda I$ is bijective for all $\lambda \in \Sigma_p(\varepsilon)$ and that its inverse $(A_p - \lambda I)^{-1}$ satisfies estimate (1.4).

Theorem 7.1 is proved, apart from the proof of Proposition 7.2. The proof of Proposition 7.2 will be given in the next subsection, due to its length.

7.1.1 Proof of Proposition 7.2

The proof of Proposition 7.2 is divided into three steps.

Step 1: First, we study the null spaces $\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ and $\mathcal{N}\left(\mathcal{T}_p(\lambda')\right)$ when $\lambda' = \ell^2 e^{i\theta}$ with $\ell \in \mathbf{Z}$:

$$\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \left\{\tilde{\varphi} \in B^{2-1/p,p}(\partial D \times S) : \widetilde{T}(\theta)\tilde{\varphi} = 0\right\},$$
$$\mathcal{N}\left(\mathcal{T}_p(\lambda')\right) = \left\{\varphi \in B^{2-1/p,p}(\partial D) : T(\lambda')\varphi = 0\right\}.$$

Since the pseudo-differential operators $\widetilde{T}(\theta)$ and $T(\lambda')$ are both *hypoelliptic*, it follows that

$$\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \left\{\tilde{\varphi} \in C^\infty(\partial D \times S) : \widetilde{T}(\theta)\tilde{\varphi} = 0\right\},$$
$$\mathcal{N}\left(\mathcal{T}_p(\lambda')\right) = \left\{\varphi \in C^\infty(\partial D) : T(\lambda')\varphi = 0\right\}.$$

Therefore, we can apply [Ta2, Proposition 8.4.6] to obtain the following most important relationship between the null spaces $\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ and $\mathcal{N}\left(\mathcal{T}_p(\lambda')\right)$ when $\lambda' = \ell^2 e^{i\theta}$ with $\ell \in \mathbf{Z}$:

Lemma 7.4. *The following two conditions are equivalent:*

(1) $\dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) < \infty$.
(2) There exists a finite subset I *of* $\mathbf{Z}$ *such that*

$$\begin{cases} \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right) < \infty & \text{if } \ell \in I, \\ \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right) = 0 & \text{if } \ell \notin I. \end{cases}$$

Moreover, in this case we have the formulas

$$\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \bigoplus_{\ell \in I} \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right) \otimes e^{i\ell y},$$
$$\dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \sum_{\ell \in I} \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right).$$

Step 2: Secondly, we study the ranges $\mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ and $\mathcal{R}\left(\mathcal{T}_p(\lambda')\right)$ when $\lambda' = \ell^2 e^{i\theta}$ with $\ell \in \mathbf{Z}$. To do this, we consider the adjoint operators $\widetilde{\mathcal{T}}_p(\theta)^*$ and $\mathcal{T}_p(\lambda')^*$ of $\widetilde{\mathcal{T}}_p(\theta)$ and $\mathcal{T}_p(\lambda')$, respectively.

The next lemma allows us to give a characterization of the adjoint operators $\widetilde{\mathcal{T}}_p(\theta)^*$ and $\mathcal{T}_p(\lambda')^*$ in terms of pseudo-differential operators (cf. [Ta2, Lemma 8.4.8]):

Lemma 7.5. *Let M be a compact smooth manifold without boundary. If T is a classical pseudo-differential operator of order m on M, we define a densely defined, closed linear operator*

$$\mathcal{T} : B^{s,p}(M) \longrightarrow B^{s-m+1,p}(M) \quad (s \in \mathbf{R})$$

as follows.

(a) The domain $\mathcal{D}(\mathcal{T})$ of $\mathcal{T}$ is the space

$$\mathcal{D}(\mathcal{T}) = \left\{\varphi \in B^{s,p}(M) : T\varphi \in B^{s-m+1,p}(M)\right\}.$$

(b) $\mathcal{T}\varphi = T\varphi$, $\varphi \in \mathcal{D}(\mathcal{T})$.

Then the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is characterized as follows:

(c) The domain $\mathcal{D}(\mathcal{T}^*)$ of $\mathcal{T}^*$ is contained in the space

$$\left\{\psi \in B^{-s+m-1,p'}(M) : T^*\psi \in B^{-s,p'}(M)\right\},$$

where $p' = p/(p-1)$ and $T^* \in L^m_{\mathrm{cl}}(M)$ is the adjoint of T.
(d) $\mathcal{T}^*\psi = T^*\psi$, $\psi \in \mathcal{D}(\mathcal{T}^*)$.

Proof. Let ψ be an arbitrary element of $\mathcal{D}(\mathcal{T}^*) \subset B^{-s+m-1,p'}(M)$, and let $\{\psi_j\}$ be a sequence in $C^\infty(M)$ such that $\psi_j \to \psi$ in $B^{-s+m-1,p'}(M)$. Then we have, for all $\varphi \in C^\infty(M) \subset \mathcal{D}(\mathcal{T})$,

$$\begin{aligned}(\mathcal{T}^*\psi, \varphi) &= (\psi, \mathcal{T}\varphi) \\ &= (\psi, T\varphi) \\ &= \lim_{j\to\infty} (\psi_j, T\varphi) \\ &= \lim_{j\to\infty} (T^*\psi_j, \varphi) \\ &= (T^*\psi, \varphi).\end{aligned}$$

This proves that

$$T^*\psi = \mathcal{T}^*\psi \in B^{-s,p'}(M).$$

The proof of Lemma 7.5 is complete. □

We remark that the pseudo-differential operators $T(\lambda)^*$ and $\widetilde{T}(\theta)^*$ also satisfy conditions (3.7a) and (3.7b) of Theorem 3.16 with $\mu := 0$, $\rho := 1$ and $\delta := 1/2$; hence they are *hypoelliptic*.

Therefore, by applying Lemma 7.5 to the operators $\widetilde{T}(\theta)$ and $T(\lambda')$ we obtain the following:

Lemma 7.6. *The null spaces $\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right)$ and $\mathcal{N}\left(\mathcal{T}_p(\lambda')^*\right)$ are characterized respectively as follows:*

$$\mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right) = \left\{\tilde{\psi} \in C^\infty(\partial D \times S) : \widetilde{T}(\theta)^*\tilde{\psi} = 0\right\}.$$
$$\mathcal{N}\left(\mathcal{T}_p(\lambda')^*\right) = \left\{\psi \in C^\infty(\partial D) : T(\lambda')^*\psi = 0\right\}.$$

By using Lemma 7.6, we find that Lemma 7.4 remains valid for the adjoint operators $\widetilde{\mathcal{T}}_p(\theta)^*$ and $\mathcal{T}_p(\lambda')^*$ (cf. [Ta2, Lemma 8.4.10]):

Lemma 7.7. *The following two conditions are equivalent:*

(1) $\dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right) < \infty$.
(2) There exists a finite subset J *of* $\mathbf{Z}$ *such that*

$$\begin{cases} \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})^*\right) < \infty & \text{if } \ell \in J, \\ \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})^*\right) = 0 & \text{if } \ell \notin J. \end{cases}$$

Moreover, in this case we have the formula

$$\dim \mathcal{N}\left(\widetilde{\mathcal{T}}_p(\theta)^*\right) = \sum_{\ell \in J} \dim \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})^*\right).$$

Hence, by combining Lemma 7.7 and the closed range theorem ([Yo, Chapter VII, Section 5, Theorem]) we obtain the most important relationship between $\operatorname{codim} \mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right)$ and $\operatorname{codim} \mathcal{R}\left(\mathcal{T}_p(\lambda')\right)$ when $\lambda' = \ell^2 e^{i\theta}$, $\ell \in \mathbf{Z}$ (cf. [Ta2, Proposition 8.4.11]):

Lemma 7.8. *The following two conditions are equivalent:*

(1) $\operatorname{codim} \mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right) < \infty$.
(2) There exists a finite subset J *of* $\mathbf{Z}$ *such that*

$$\begin{cases} \operatorname{codim} \mathcal{R}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right) < \infty & \text{if } \ell \in J, \\ \operatorname{codim} \mathcal{N}\left(\mathcal{T}_p(\ell^2 e^{i\theta})\right) = 0 & \text{if } \ell \notin J. \end{cases}$$

Moreover, in this case we have the formula

$$\operatorname{codim} \mathcal{R}\left(\widetilde{\mathcal{T}}_p(\theta)\right) = \sum_{\ell \in J} \operatorname{codim} \mathcal{R}\left(\mathcal{T}_p(\ell^2 e^{i\theta})^*\right).$$

Step 3: Proposition 7.2 is an immediate consequence of Lemmas 7.4 and 7.8, with $K := I \cup J$.

The proof of Proposition 7.2 is now complete. □

Summing up, we have proved Theorem 7.1 and hence part (i) of Theorem 1.2. □

7.2 Proof of Theorem 1.2, Part (ii)

Part (ii) of Theorem 1.2 may be proved as follows. Theorem 7.1 asserts that, for sufficiently large $\mu_\varepsilon > 0$, the operator $A_p - \mu_\varepsilon I$ satisfies condition (2.1) (see Figure 7.1). Thus, by applying Theorem 2.2 (and Remark 2.1) to the operator $A_p - \mu_\varepsilon I$ we obtain part (ii) of Theorem 1.2:

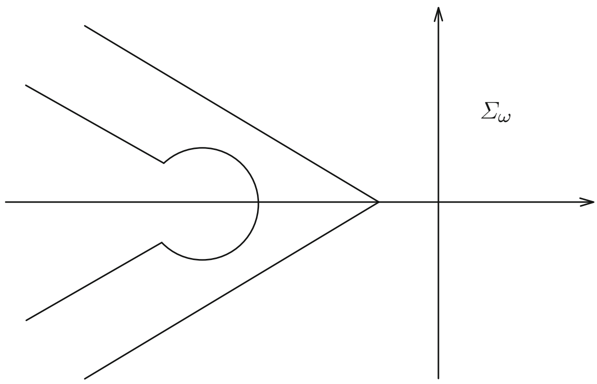

Fig. 7.1.

Theorem 7.9. *Assume that conditions (A) and (B) are satisfied. Then the operator* A_p *generates a semigroup* U_z *on* $L^p(D)$ *which is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0,\ |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$*, and enjoys the following three properties:*

(a) The operators A_pU_z *and* $\frac{dU_z}{dz}$ *are bounded operators on* $L^p(D)$ *for each* $z \in \Delta_\varepsilon$*, and satisfy the relation*

$$\frac{dU_z}{dz} = A_pU_z \quad \text{for all } z \in \Delta_\varepsilon.$$

(b) For each $0 < \varepsilon < \pi/2$*, there exist positive constants* $\widetilde{M}_0(\varepsilon)$*,* $\widetilde{M}_1(\varepsilon)$ *and* μ_ε *such that*

$$\|U_z\| \leq \widetilde{M}_0(\varepsilon)e^{\mu_\varepsilon \cdot \mathrm{Re}\, z} \qquad \text{for all } z \in \Delta_\varepsilon,$$

$$\|A_pU_z\| \leq \frac{\widetilde{M}_1(\varepsilon)}{|z|}e^{\mu_\varepsilon \cdot \mathrm{Re}\, z} \quad \text{for all } z \in \Delta_\varepsilon.$$

(c) For each $u_0 \in L^p(D)$*, we have, as* $z \to 0, z \in \Delta_\varepsilon$*,*

$$U_z u_0 \longrightarrow u_0 \quad \text{in } L^p(D).$$

The proof of Theorem 1.2 is now complete. □

8

Proof of Theorem 1.3, Part (i)

This Chapter 8 and the next Chapter 9 are devoted to the proof of Theorem 1.3 and Theorem 1.4. In this chapter we prove part (i) of Theorem 1.3. In the proof we make use of Sobolev's imbedding theorems (Theorems 8.1 and 8.2) and a λ-*dependent localization* argument due to Masuda [Ma] (cf. Lemma 8.4) in order to adjust estimate

$$\left\|(A_p - \lambda I)^{-1}\right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon) \tag{1.4}$$

to obtain the desired estimate

$$\|(\mathfrak{A} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon). \tag{1.6}$$

Here we recall that

$$\mathcal{D}(A_p) = \left\{u \in H^{2,p}(D) = W^{2,p}(D) : Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0\right\}. \tag{1.3}$$

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ Lu = 0\right\}. \tag{1.5}$$

8.1 The Space $C_0(\overline{D} \setminus M)$

First, we consider a one-point compactification $K_\partial = K \cup \{\partial\}$ of the space $K = \overline{D} \setminus M$.

We say that two points x and y of $\overline{D}$ are equivalent modulo M if $x = y$ or $x, y \in M$; we then write $x \sim y$. It is easy to verify that this relation $\sim$ enjoys the so-called equivalence laws. We denote by $\overline{D}/M$ the totality of equivalence classes modulo M. On the set $\overline{D}/M$ we define the quotient topology induced by the projection

K. Taira, *Boundary Value Problems and Markov Processes*,
Lecture Notes in Mathematics 1499, DOI 10.1007/978-3-642-01677-6_8,

$$q : \overline{D} \longrightarrow \overline{D}/M.$$

Namely, a subset O of $\overline{D}/M$ is defined to be open if and only if the inverse image $q^{-1}(O)$ of O is open in $\overline{D}$. It is easy to see that the topological space $\overline{D}/M$ is a *one-point compactification* of the space $\overline{D} \setminus M$ and that the *point at infinity* ∂ corresponds to the set M (see Figure 8.1):

$$\begin{cases} K_\partial := \overline{D}/M, \\ \partial := M. \end{cases}$$

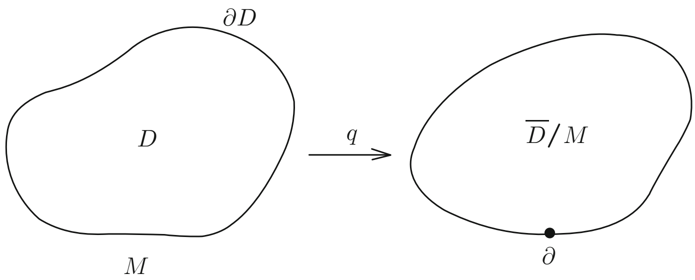

Fig. 8.1.

Furthermore, we obtain the following two assertions:

(i) If $\tilde{u}$ is a continuous function defined on K_∂, then the function $\tilde{u} \circ q$ is continuous on $\overline{D}$ and constant on M.
(ii) Conversely, if u is a continuous function defined on $\overline{D}$ and constant on M, then it defines a continuous function $\tilde{u}$ on K_∂.

In other words, we have the following isomorphism:

$$C(K_\partial) \cong \left\{ u \in C(\overline{D}) : u(x) \text{ is constant on } M \right\}. \tag{8.1}$$

Now we introduce a closed subspace of $C(K_\partial)$ as in Subsection 2.2.1:

$$C_0(K) = \{ u \in C(K_\partial) : u(\partial) = 0 \}.$$

Then we have, by assertion (8.1),

$$C_0(K) \cong C_0(\overline{D} \setminus M) = \left\{ u \in C(\overline{D}) : u(x) = 0 \text{ on } M \right\}. \tag{8.2}$$

8.2 Sobolev's Imbedding Theorems

It is the imbedding characteristics of Sobolev spaces of L^p type that render these spaces so useful in the study of partial differential equations. We need the following imbedding properties of Sobolev spaces:

Theorem 8.1 (Sobolev). *Let D be a bounded domain in the Euclidean space $\mathbf{R}^N$ with boundary ∂D of class C^2. Then we have the following two assertions:*

(i) If $1 \leq p < N$, we have the continuous injection

$$W^{2,p}(D) \subset W^{1,q}(D) \quad \text{for} \;\; \tfrac{1}{p} - \tfrac{1}{N} \leq \tfrac{1}{q} \leq \tfrac{1}{p}.$$

(ii) If $N/2 < p < \infty$, $p \neq N$, we have the continuous injection

$$W^{2,p}(D) \subset C^{\nu}(\overline{D}) \quad \text{for} \;\; 0 < \nu \leq 2 - \tfrac{N}{p}.$$

Theorem 8.2 (Gagliardo–Nirenberg). *Let D be a bounded domain in $\mathbf{R}^N$ with boundary of class C^2, and $1 \leq p, r \leq \infty$. Then we have the following assertions:*

(i) If $p \neq N$ and if

$$\frac{1}{q} = \frac{1}{N} + \theta \left(\frac{1}{p} - \frac{2}{N} \right) + (1-\theta)\frac{1}{r} \quad \text{for} \;\; \tfrac{1}{2} \leq \theta \leq 1,$$

then we have, for all $u \in W^{2,p}(D) \cap L^r(D)$,

$$\|u\|_{1,q} \leq C_1 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta},$$

with a positive constant $C_1 = C_1(D, p, r, \theta)$.

(ii) If $N/2 < p < \infty$, $p \neq N$ and if

$$0 \leq \nu < \theta \left(2 - \frac{N}{p} \right) - (1-\theta)\frac{N}{r},$$

then we have, for all $u \in W^{2,p}(D) \cap L^r(D)$,

$$\|u\|_{C^{\nu}(\overline{D})} \leq C_2 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta}, \tag{8.3}$$

with a positive constant $C_2 = C_2(D, p, r, \theta)$.

For a proof of Theorem 8.1, see Adams–Fournier [AF, Theorem 5.4] and for a proof of Theorem 8.2, see Friedman [Fr1, Part I, Theorem 10.1], and also Taira [Ta4].

8.3 Proof of Part (i) of Theorem 1.3

The proof is carried out in a chain of auxiliary lemmas.

Step (I): We begin with a version of estimate (7.1):

Lemma 8.3. *Let $N < p < \infty$. Assume that the following conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ *on* ∂D.
(B) $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then, for every $\varepsilon > 0$*, there exists a positive constant* $r_p(\varepsilon)$ *such that if* $\lambda = r^2 e^{i\theta}$ *with* $r \geq r_p(\varepsilon)$ *and* $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$*, we have, for all* $u \in \mathcal{D}(A_p)$*,*

$$|\lambda|^{1/2}|u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \leq C_p(\varepsilon)|\lambda|^{N/2p}\|(A-\lambda)u\|_p, \tag{8.4}$$

with a positive constant $C_p(\varepsilon)$*. Here*

$$\mathcal{D}(A_p) = \left\{u \in H^{2,p}(D) = W^{2,p}(D) : Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0\right\}.$$

Proof. First, by applying Theorem 8.2 with $p := r > N$, $\theta := N/p$ and $\nu := 0$ we obtain from the Gagliardo–Nirenberg inequality (8.3) that

$$|u|_{C(\overline{D})} \leq C|u|_{1,p}^{N/p}\|u\|_p^{1-N/p}. \tag{8.5}$$

Here and in the following the letter C denotes a generic positive constant depending on p and ε, but independent of u and λ.

Combining inequality (7.1) with inequality (8.5), we find that

$$\begin{aligned} |u|_{C(\overline{D})} &\leq C\left(|\lambda|^{-1/2}\|(A-\lambda)u\|_p\right)^{N/p}\left(|\lambda|^{-1}\|(A-\lambda)u\|_p\right)^{1-N/p} \\ &= C|\lambda|^{-1+N/2p}\|(A-\lambda)u\|_p, \end{aligned}$$

so that

$$|\lambda| \cdot |u|_{C(\overline{D})} \leq C|\lambda|^{N/2p}\|(A-\lambda)u\|_p \quad \text{for all } u \in \mathcal{D}(A_p). \tag{8.6}$$

Similarly, by applying inequality (8.5) to the functions $D_i u \in W^{1,p}(D)$, $1 \leq i \leq n$, we obtain that

$$\begin{aligned} |D_i u|_{C(\overline{D})} &\leq C|D_i u|_{1,p}^{N/p}\|D_i u\|_p^{1-N/p} \\ &\leq C|u|_{2,p}^{N/p}|u|_{1,p}^{1-N/p} \\ &\leq C\left(\|(A-\lambda)u\|_p\right)^{N/p}\left(|\lambda|^{-1/2}\|(A-\lambda)u\|_p\right)^{1-N/p} \\ &= C|\lambda|^{-1/2+N/2p}\|(A-\lambda)u\|_p. \end{aligned}$$

This proves that

$$|\lambda|^{1/2}|u|_{C^1(\overline{D})} \leq C|\lambda|^{N/2p}\|(A-\lambda)u\|_p \quad \text{for all } u \in \mathcal{D}(A_p). \tag{8.7}$$

Therefore, the desired inequality (8.4) follows by combining inequalities (8.6) and (8.7). □

The next lemma proves estimate (1.6):

Lemma 8.4. *Assume that conditions (A) and (B) are satisfied. Then, for every* $\varepsilon > 0$, *there exists a positive constant* $r(\varepsilon)$ *such that if* $\lambda = r^2 e^{i\theta}$ *with* $r \geq r(\varepsilon)$ *and* $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$, *we have, for all* $u \in \mathcal{D}(\mathfrak{A})$,

$$|\lambda|^{1/2}|u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \leq c(\varepsilon)|(A - \lambda)u|_{C(\overline{D})}, \tag{8.8}$$

with a positive constant $c(\varepsilon)$. *Here*

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ Lu = 0\right\}.$$

Proof. We shall make use of a λ-*dependent localization* argument due to Masuda [Ma] in order to adjust the term $\|(A - \lambda)u\|_p$ in inequality (8.4) to obtain inequality (8.8).

First, we remark that

$$\mathfrak{A} \subset A_p \quad \text{for all} \quad 1 < p < \infty.$$

Indeed, since we have, for any $u \in \mathcal{D}(\mathfrak{A})$,

$$u \in C(\overline{D}) \subset L^p(D),\ Au \in C(\overline{D}) \subset L^p(D) \quad \text{and} \quad Lu = 0,$$

it follows from an application of Theorem 4.9 and Lemma 5.1 that

$$u \in W^{2,p}(D).$$

(1) Let x_0 be an arbitrary point of the closure $\overline{D} = D \cup \partial D$.

If x_0' is a *boundary* point and if χ is a smooth coordinate transformation such that χ maps $B(x_0, \eta_0) \cap D$ into $B(0, \delta) \cap \mathbf{R}^N_+$ and flattens a part of the boundary ∂D into the plane $x_N = 0$ (see Figure 8.2), then we let

$$\begin{aligned} G_0 &= B(x_0', \eta_0) \cap D, \\ G' &= B(x_0', \eta) \cap D,\ 0 < \eta < \eta_0, \\ G'' &= B(x_0', \eta/2) \cap D,\ 0 < \eta < \eta_0. \end{aligned}$$

Here $B(x, \eta)$ denotes the open ball of radius η about x.

Similarly, if x_0 is an *interior* point and if χ is a smooth coordinate transformation such that χ maps $B(x_0, \eta_0)$ into $B(0, \delta)$, then we let (see Figure 8.3)

$$\begin{aligned} G_0 &= B(x_0, \eta_0), \\ G' &= B(x_0, \eta),\ 0 < \eta < \eta_0, \\ G'' &= B(x_0, \eta/2),\ 0 < \eta < \eta_0. \end{aligned}$$

(2) Now we take a function $\theta(t)$ in $C_0^\infty(\mathbf{R})$ such that $\theta(t)$ equals one near the origin, and define

$$\varphi(x) = \theta(|x'|^2)\,\theta(x_N), \quad x = (x', x_N).$$

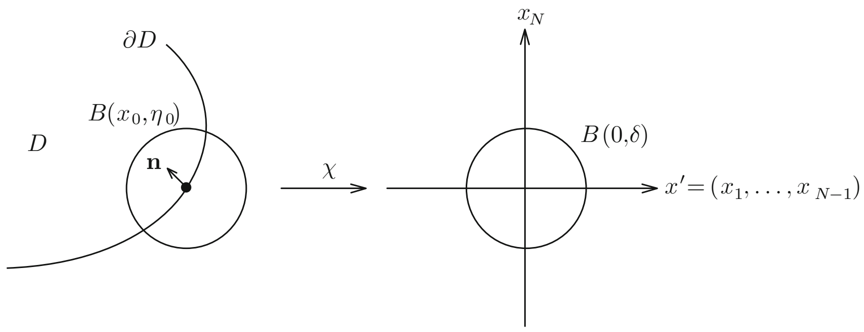

Fig. 8.2.

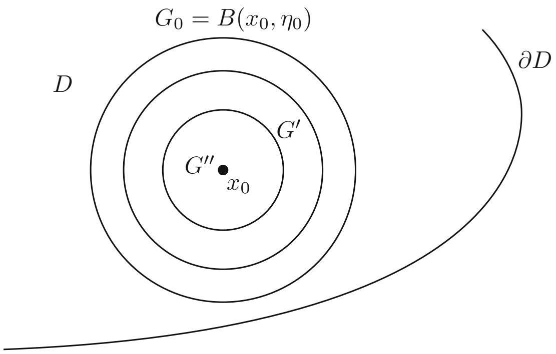

Fig. 8.3.

Here we may assume that the function $\varphi(x)$ is chosen so that

$$\begin{cases} \operatorname{supp}\varphi \subset B(0,1), \\ \varphi(x) = 1 \text{ on } B(0,1/2). \end{cases}$$

We introduce a localizing function

$$\varphi_0(x,\eta) \equiv \varphi\left(\frac{x-x_0}{\eta}\right) = \theta\left(\frac{|x'-x_0'|^2}{\eta^2}\right)\theta\left(\frac{x_N-t}{\eta}\right), \quad x_0 = (x_0', t) \in \overline{D}.$$

We remark that

$$\begin{cases} \operatorname{supp}\varphi_0 \subset B(x_0,\eta), \\ \varphi_0(x,\eta) = 1 \text{ on } B(x_0,\eta/2). \end{cases}$$

Then we have the following:

Claim 8.5. *If $u \in \mathcal{D}(\mathfrak{A})$, then it follows that $\varphi_0(x,\eta)u \in \mathcal{D}(A_p)$ for all $1 < p < \infty$.*

Proof. (i) First, we recall that

$$u \in W^{2,p}(D) \quad \text{for all} \quad 1 < p < \infty.$$

Hence we have the assertion

$$\varphi_0(x, \eta)u \in W^{2,p}(D).$$

(ii) Secondly, it is easy to verify (see Figure 8.4) that the function $\varphi_0(x, \eta)u$, $x \in \overline{D}$, satisfies the boundary condition

$$L(\varphi_0(x, \eta)u) = 0 \quad \text{on } \partial D.$$

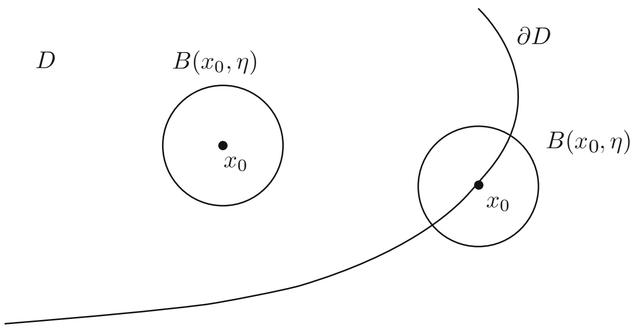

Fig. 8.4.

Indeed, this is obvious if we have the condition

$$\operatorname{supp}(\varphi_0(x, \eta)u) \subset B(x_0, \eta), \quad x_0 \in D.$$

Moreover, if we have the condition

$$\operatorname{supp}(\varphi_0(x, \eta)u) \subset B(x_0, \eta) \cap \overline{D}, \quad x_0 \in \partial D,$$

then it follows that

$$\frac{\partial}{\partial x_N}(\varphi_0(x, \eta))\bigg|_{x_N=0} = \frac{1}{\eta}\,\theta'(0) \cdot \theta\left(\frac{|x' - x_0'|^2}{\eta^2}\right) = 0,$$

since $\theta'(0) = 0$. This proves that

$$\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x, \eta)) = 0 \quad \text{on } \partial D.$$

Therefore, we have the assertion

$$\begin{aligned} L(\varphi_0(x,\eta)u) &= \mu(x')\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x,\eta)u) + \gamma(x')\varphi_0(x,\eta)u \\ &= \varphi_0(x,\eta)(Lu) + \mu(x')\left(\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x,\eta))\right)u \\ &= 0 \quad \text{on } \partial D, \end{aligned}$$

since $Lu = 0$ on ∂D.

Summing up, we have proved that

$$\varphi_0(x,\eta)u \in \mathcal{D}(A_p) \quad \text{for all } 1 < p < \infty.$$

The proof of Claim 8.5 is complete. □

(3) Now we take a positive number p such that

$$N < p < \infty.$$

Then, by Claim 8.5 we can apply inequality (8.4) to the function $\varphi_0(x,\eta)u$, $u \in \mathcal{D}(\mathfrak{A})$, to obtain that

$$\begin{aligned} &|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda|\cdot|u|_{C(\overline{G''})} \\ &\le |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{G'})} + |\lambda|\cdot\left|\varphi_0(x,\eta)u\right|_{C(\overline{G'})} \\ &= |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{D})} + |\lambda|\cdot\left|\varphi_0(x,\eta)u\right|_{C(\overline{D})} \\ &\le C|\lambda|^{N/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(D)} \\ &= C|\lambda|^{N/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(G')}, \quad 0 < \eta < \eta_0, \end{aligned} \tag{8.9}$$

since we have the assertions

$$\begin{cases} \varphi_0(x,\eta) = 1 \quad \text{on } G'', \\ \operatorname{supp}(\varphi_0(x,\eta)u) \subset \overline{G'}. \end{cases}$$

However, we have the formula

$$(A-\lambda)(\varphi_0(x,\eta)u) = \varphi_0(x,\eta)\left((A-\lambda)u\right) + [A,\varphi_0(x,\eta)]u, \tag{8.10}$$

where $[A,\varphi_0(x,\eta)]$ is the commutator of A and $\varphi_0(x,\eta)$ defined by the formula

$$\begin{aligned} [A,\varphi_0(x,\eta)]u &= A(\varphi_0(x,\eta)u) - \varphi_0(x,\eta)Au \\ &= 2\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial\varphi_0}{\partial x_i}\frac{\partial u}{\partial x_j} \\ &\quad + \left(\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2\varphi_0}{\partial x_i\partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial\varphi_0}{\partial x_i}\right)u. \end{aligned} \tag{8.11}$$

Here we need the following elementary inequality:

Claim 8.6. *We have, for all* $v \in C^j(\overline{G'})$, $j = 0, 1, 2$,

$$\|v\|_{W^{j,p}(G')} \leq |G'|^{1/p} \, \|v\|_{C^j(\overline{G'})},$$

where $|G'|$ *denotes the measure of* G'.

Proof. It suffices to note that we have, for all $w \in C(\overline{G'})$,

$$\int_{G'} |w(x)|^p \, dx \leq |G'| \, |w|^p_{C(\overline{G'})}.$$

This proves Claim 8.6. □

Since we have (see Figure 8.3), for some positive constant c,

$$|G'| \leq |B(x_0, \eta)| \leq c\eta^N,$$

it follows from an application of Claim 8.6 that

$$\|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} \leq c^{1/p}\eta^{N/p}|(A-\lambda)u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0. \tag{8.12}$$

Furthermore, we remark that

$$|D^\alpha \varphi_0(x,\eta)| = O\left(\eta^{-|\alpha|}\right) \quad \text{as } \eta \downarrow 0.$$

Hence it follows from an application of Claim 8.6 that

$$\left\| \frac{\partial \varphi_0}{\partial x_i} \frac{\partial u}{\partial x_j} \right\|_{L^p(G')} \leq \frac{C}{\eta} |u|_{1,p,G'} \leq C\eta^{-1+N/p}|u|_{C^1(\overline{G'})}, \quad 0 < \eta < \eta_0, \tag{8.13}$$

$$\left\| \frac{\partial^2 \varphi_0}{\partial x_i \partial x_j} u \right\|_{L^p(G')} \leq \frac{C}{\eta^2} |u|_{L^p(G')} \leq C\eta^{-2+N/p}|u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0, \tag{8.14}$$

$$\left\| \frac{\partial \varphi_0}{\partial x_i} u \right\|_{L^p(G')} \leq \frac{C}{\eta} |u|_{L^p(G')} \leq C\eta^{-1+N/p}|u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0. \tag{8.15}$$

By using inequalities (8.13), (8.14) and (8.15), we obtain from formula (8.11) that

$$\begin{aligned} &\|[A, \varphi_0(x,\eta)]u\|_{L^p(G')} \\ &\leq C\left(\eta^{-1+N/p}|u|_{C^1(\overline{G'})} + \eta^{-2+N/p}|u|_{C(\overline{G'})} + \eta^{-1+N/p}|u|_{C(\overline{G'})}\right) \\ &\leq C\left(\eta^{-1+N/p}|u|_{C^1(\overline{D})} + \eta^{-2+N/p}|u|_{C(\overline{D})}\right), \quad 0 < \eta < \eta_0. \end{aligned} \tag{8.16}$$

In view of formula (8.10), it follows from inequalities (8.12) and (8.16) that

$$\begin{aligned} &\|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\ &\leq \|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} + \|[A, \varphi_0(x,\eta)]u\|_{L^p(G')} \\ &\leq C\eta^{N/p}\left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})}\right), \\ &\qquad 0 < \eta < \eta_0. \end{aligned} \tag{8.17}$$

Therefore, by combining inequalities (8.9) and (8.17) we obtain that

$$\begin{aligned}
&|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda| \cdot |u|_{C(\overline{G''})} \\
&\leq C|\lambda|^{N/2p} \, \|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\
&\leq C|\lambda|^{N/2p} \, \eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{G'})} + \eta^{-2}|u|_{C(\overline{G'})} \right) \\
&\leq C|\lambda|^{N/2p} \, \eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{D})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})} \right), \\
&\qquad 0 < \eta < \eta_0. \qquad (8.18)
\end{aligned}$$

We remark (see Figure 8.5) that the closure $\overline{D} = D \cup \partial D$ can be covered by a finite number of sets of the forms

$$B(x_0', \eta/2) \cap \overline{D}, \quad x_0' \in \partial D,$$

and

$$B(x_0, \eta/2), \quad x_0 \in D.$$

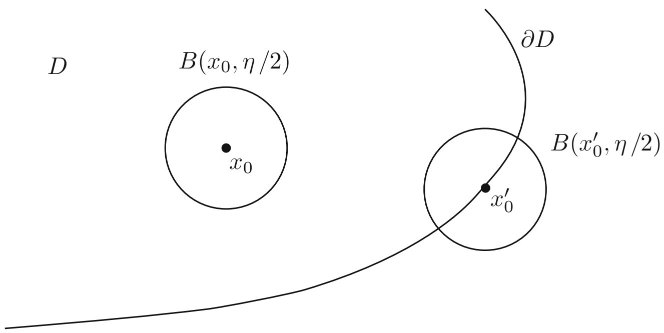

Fig. 8.5.

Hence, by taking the supremum of inequality (8.18) over $x \in \overline{D}$ we find that

$$\begin{aligned}
&|\lambda|^{1/2}|u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \\
&\leq C|\lambda|^{N/2p}\eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{D})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})} \right), \\
&\qquad 0 < \eta < \eta_0. \qquad (8.19)
\end{aligned}$$

(4) We now choose the localization parameter η. We let

$$\eta = \frac{\eta_0}{|\lambda|^{1/2}} K,$$

where K is a positive constant (to be chosen later) satisfying the condition

$$0 < \eta = \frac{\eta_0}{|\lambda|^{1/2}} K < \eta_0,$$

that is,

$$0 < K < |\lambda|^{1/2}.$$

Then it follows from inequality (8.19) that

$$\begin{aligned} &|\lambda|^{1/2} |u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \\ &\leq C\, \eta_0^{N/p} K^{N/p} |(A-\lambda)u|_{C(\overline{D})} + \left(C\, \eta_0^{N/p-1} K^{-1+N/p} \right) |\lambda|^{1/2} \cdot |u|_{C^1(\overline{D})} \\ &\quad + \left(C\, \eta_0^{N/p-2} K^{-2+N/p} \right) |\lambda| \cdot |u|_{C(\overline{D})} \quad \text{for all } u \in \mathcal{D}(\mathfrak{A}). \end{aligned} \tag{8.20}$$

However, since the exponents $-1 + N/p$ and $-2 + N/p$ are negative for $N < p < \infty$, we can choose the constant K so large that

$$C\, \eta_0^{N/p-1} K^{-1+N/p} < 1,$$

and

$$C\, \eta_0^{N/p-2} K^{-2+N/p} < 1.$$

Then the desired inequality (8.8) follows from inequality (8.20).

The proof of Lemma 8.4 is complete. □

Step (II): The next lemma, together with Lemma 8.4, proves that the resolvent set of $\mathfrak{A}$ contains the set

$$\Sigma(\varepsilon) = \left\{ \lambda = r^2 e^{i\theta} : r \geq r(\varepsilon),\ \ -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon \right\},$$

that is, the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ exists for all $\lambda \in \Sigma(\varepsilon)$.

Lemma 8.7. *If $\lambda \in \Sigma(\varepsilon)$, then, for any $f \in C_0(\overline{D} \setminus M)$, there exists a unique function $u \in \mathcal{D}(\mathfrak{A})$ such that $(\mathfrak{A} - \lambda I)u = f$.*

Proof. Since we have the assertion

$$f \in C_0(\overline{D} \setminus M) \subset L^p(D) \quad \text{for all} \quad 1 < p < \infty,$$

it follows from an application of Theorem 1.2 that if $\lambda \in \Sigma(\varepsilon)$ there exists a unique function $u \in W^{2,p}(D)$ such that

$$(A - \lambda)u = f \quad \text{in } D, \tag{8.21}$$

and

$$Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \quad \text{on } \partial D. \tag{8.22}$$

However, part (ii) of Theorem 8.1 asserts that

$$u \in W^{2,p}(D) \subset C^{2-N/p}(\overline{D}) \subset C^1(\overline{D}) \quad \text{if} \quad N < p < \infty.$$

Hence we have, by formula (8.22) and condition (B),

$$u = 0 \text{ on } M = \{x' \in \partial D : \mu(x') = 0\},$$

so that

$$u \in C_0(\overline{D} \setminus M).$$

Furthermore, in view of formula (8.21) it follows that

$$Au = f + \lambda u \in C_0(\overline{D} \setminus M).$$

Summing up, we have proved that

$$\begin{cases} u \in \mathcal{D}(\mathfrak{A}), \\ (\mathfrak{A} - \lambda I)u = f. \end{cases}$$

Now the proof of part (i) of Theorem 1.3 is complete. $\square$

9

Proof of Theorem 1.3, Part (ii)

In this chapter we prove Theorem 1.4 and part (ii) of Theorem 1.3. This chapter is the heart of the subject. General existence theorems for Feller semigroups are formulated in terms of elliptic boundary value problems with spectral parameter (Theorem 9.12). First, we study Feller semigroups with reflecting barrier (Theorem 9.14) and then, by using these Feller semigroups we construct Feller semigroups corresponding to such a diffusion phenomenon that either absorption or reflection phenomenon occurs at each point of the boundary (Theorem 9.18). Our proof is based on the generation theorems of Feller semigroups discussed in Section 2.2.

9.1 General Existence Theorem for Feller Semigroups

The purpose of this section is to give a general existence theorem for Feller semigroups in terms of boundary value problems (Theorem 9.12), following Taira [Ta2, Section 9.6] (cf. [BCP], [SU], [Ta3]).

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$A = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a second-order, *elliptic* differential operator with real coefficients such that:

(1) $a^{ij} \in C^\infty(\overline{D})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$ and $1 \leq i,j \leq N$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{D} \times \mathbf{R}^N.$$

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(2) $b^i \in C^\infty(\overline{D})$ for all $1 \leq i \leq N$.
(3) $c \in C^\infty(\overline{D})$ and $c(x) \leq 0$ on $\overline{D}$.

The differential operator A describes analytically a strong Markov process with continuous paths in the interior D such as Brownian motion (see Figure 1.4). The functions $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are called the diffusion coefficients, the drift coefficients and the termination coefficient, respectively.

Let L be a first-order boundary condition such that

$$Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u,$$

where:

(4) $\mu \in C^\infty(\partial D)$ and $\mu(x') \geq 0$ on ∂D.
(5) $\gamma \in C^\infty(\partial D)$ and $\gamma(x') \leq 0$ on ∂D.
(6) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D (see Figure 1.1).

The boundary condition L is called a first-order *Ventcel' boundary condition* (cf. [We]). Its terms $\mu(x')(\partial u)/(\partial \mathbf{n})$ and $\gamma(x')u$ are supposed to correspond to the reflection and absorption phenomena, respectively (see Figure 1.5).

We are interested in the following problem:

Problem. Given analytic data (A, L), can we construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ is characterized by (A, L) ?

First, we consider the following Dirichlet problem: Given functions $f(x)$ and $\varphi(x')$ defined in D and on ∂D, respectively, find a function $u(x)$ in D such that

$$\begin{cases} Au = f & \text{in } D, \\ u = \varphi & \text{on } \partial D. \end{cases} \tag{9.1}$$

The next theorem summarizes the basic facts about the Dirichlet problem in the framework of *Hölder spaces* (cf. [GT]):

Theorem 9.1. *(i) (*Existence and Uniqueness*) If $f \in C^\theta(D)$ with $0 < \theta < 1$ and if $\varphi \in C(\partial D)$, then problem (9.1) has a unique solution $u(x)$ in $C(\overline{D}) \cap C^{2+\theta}(D)$.*

*(ii) (*Interior Regularity*) If $u \in C^2(D)$ and $Au = f \in C^{k+\theta}(D)$ for some non-negative integer k, then it follows that $u \in C^{k+2+\theta}(D)$.*

*(iii) (*Global Regularity*) If $f \in C^{k+\theta}(\overline{D})$ and $\varphi \in C^{k+2+\theta}(\partial D)$ for some non-negative integer k, then a solution $u \in C(\overline{D}) \cap C^2(D)$ of problem (9.1) belongs to the space $C^{k+2+\theta}(\overline{D})$.*

Next we consider the following Dirichlet problem with spectral parameter: For given functions $f(x)$ and $\varphi(x')$ defined in D and on ∂D, respectively, find a function $u(x)$ in D such that

$$\begin{cases} (\alpha - A)u = f & \text{in } D, \\ u = \varphi & \text{on } \partial D, \end{cases} \tag{9.2}$$

where α is a positive parameter.

By applying Theorem 9.1 with $A := A - \alpha$, we obtain that problem (9.2) has a unique solution $u(x)$ in $C^{2+\theta}(\overline{D})$ for any $f \in C^{\theta}(\overline{D})$ and any $\varphi \in C^{2+\theta}(\partial D)$ with $0 < \theta < 1$. Therefore, we can introduce linear operators

$$G_\alpha^0 : C^\theta(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D}),$$

and

$$H_\alpha : C^{2+\theta}(\partial D) \longrightarrow C^{2+\theta}(\overline{D})$$

as follows.

(a) For any $f \in C^\theta(\overline{D})$, the function $G_\alpha^0 f \in C^{2+\theta}(\overline{D})$ is the unique solution of the problem

$$\begin{cases} (\alpha - A)G_\alpha^0 f = f & \text{in } D, \\ G_\alpha^0 f = 0 & \text{on } \partial D. \end{cases} \tag{9.3}$$

(b) For any $\varphi \in C^{2+\theta}(\partial D)$, the function $H_\alpha \varphi \in C^{2+\theta}(\overline{D})$ is the unique solution of the problem

$$\begin{cases} (\alpha - A)H_\alpha \varphi = 0 & \text{in } D, \\ H_\alpha \varphi = \varphi & \text{on } \partial D. \end{cases} \tag{9.4}$$

The operator G_α^0 is called the *Green operator* and the operator H_α is called the *harmonic operator*, respectively.

Then we have the following result:

Lemma 9.2. *The operator G_α^0, $\alpha > 0$, considered from $C(\overline{D})$ into itself, is non-negative and continuous (bounded) with norm*

$$\left\|G_\alpha^0\right\| = \left\|G_\alpha^0 1\right\|_\infty = \max_{x \in \overline{D}} G_\alpha^0 1(x).$$

Proof. Let $f(x)$ be an arbitrary function in $C^\theta(\overline{D})$ such that $f(x) \geq 0$ on $\overline{D}$. Then, by applying the weak maximum principle (see Theorem A.1) with $A := A - \alpha$ to the function $-G_\alpha^0 f$ we obtain from formula (9.3) that

$$G_\alpha^0 f \geq 0 \quad \text{on } \overline{D}.$$

This proves the non-negativity of G_α^0.

Since G_α^0 is non-negative, we have, for all $f \in C^\theta(\overline{D})$,

$$-G_\alpha^0 \|f\|_\infty \leq G_\alpha^0 f \leq G_\alpha^0 \|f\|_\infty \quad \text{on } \overline{D}.$$

This implies the continuity of G_α^0 with norm

$$\|G_\alpha^0\| = \|G_\alpha^0 1\|_\infty.$$

The proof of Lemma 9.2 is complete. $\square$

Similarly, we have the following result:

Lemma 9.3. *The operator H_α, $\alpha > 0$, considered from $C(\partial D)$ into $C(\overline{D})$, is non-negative and continuous (bounded) with norm*

$$\|H_\alpha\| = \|H_\alpha 1\|_\infty = \max_{x \in \overline{D}} H_\alpha 1(x).$$

More precisely, we have the following fundamental results:

Theorem 9.4. *(i) (a) The operator G_α^0, $\alpha > 0$, can be uniquely extended to a non-negative, bounded linear operator on $C(\overline{D})$ into itself, denoted again by G_α^0, with norm*

$$\left\|G_\alpha^0\right\| = \left\|G_\alpha^0 1\right\|_\infty \leq \frac{1}{\alpha}. \tag{9.5}$$

(b) For any $f \in C(\overline{D})$, we have the assertion

$$G_\alpha^0 f = 0 \quad \text{on } \partial D.$$

(c) For all α, $\beta > 0$, the resolvent equation *holds:*

$$G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0\, G_\beta^0 f = 0, \quad f \in C(\overline{D}). \tag{9.6}$$

(d) For any $f \in C(\overline{D})$, we have the assertion

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 f(x) = f(x) \quad \text{for all } x \in D. \tag{9.7}$$

Furthermore, if $f(x') = 0$ on ∂D, then this convergence is uniform in $x \in \overline{D}$, that is, we have the assertion

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 f = f \quad \text{in } C(\overline{D}). \tag{9.8}$$

(e) The operator G_α^0 maps $C^{k+\theta}(\overline{D})$ into $C^{k+2+\theta}(\overline{D})$ for any non-negative integer k.

(ii) (a′) The operator H_α, $\alpha > 0$, can be uniquely extended to a non-negative, bounded linear operator on $C(\partial D)$ into $C(\overline{D})$, denoted again by H_α, with norm $\|H_\alpha\| = 1$.
(b′) For any $\varphi \in C(\partial D)$, we have the assertion

$$H_\alpha \varphi = \varphi \quad \text{on } \partial D.$$

(c′) For all α, $\beta > 0$, we have the equation

$$H_\alpha \varphi - H_\beta \varphi + (\alpha - \beta) G_\alpha^0\, H_\beta \varphi = 0, \quad \varphi \in C(\partial D). \tag{9.9}$$

(d′) The operator H_α maps $C^{k+2+\theta}(\partial D)$ into $C^{k+2+\theta}(\overline{D})$ for any non-negative integer k.

Proof. (i) (a) By making use of Friedrichs' mollifiers, we find that the Hölder space $C^\theta(\overline{D})$ is dense in $C(\overline{D})$ and further that non-negative functions can be approximated by non-negative smooth functions. Hence, by Lemma 9.2 it follows that the operator $G_\alpha^0 : C^\theta(\overline{D}) \to C^{2+\theta}(\overline{D})$ can be uniquely extended to a non-negative, bounded linear operator $G_\alpha^0 : C(\overline{D}) \to C(\overline{D})$ with norm $\|G_\alpha^0\| = \|G_\alpha^0 1\|_\infty$.

Furthermore, since the function $G_\alpha^0 1$ satisfies the conditions

$$\begin{cases} (A-\alpha)G_\alpha^0 1 = -1 & \text{in } D, \\ G_\alpha^0 1 = 0 & \text{on } \partial D, \end{cases}$$

by applying Theorem A.2 with $A := A - \alpha$ we obtain that

$$\|G_\alpha^0\| = \|G_\alpha^0 1\|_\infty \leq \frac{1}{\alpha}.$$

(b) This assertion follows from formula (9.3), since the space $C^\theta(\overline{D})$ is dense in $C(\overline{D})$ and since the operator $G_\alpha^0 : C(\overline{D}) \to C(\overline{D})$ is bounded.

(c) We find from the uniqueness theorem for problem (9.2) (Theorem 9.1) that equation (9.6) holds true for all $f \in C^\theta(\overline{D})$. Indeed, it suffices to note that the function

$$v = G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0 \, G_\beta^0 f \in C^{2+\theta}(\overline{D})$$

satisfies the conditions

$$\begin{cases} (\alpha - A)v = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases}$$

so that

$$v = 0 \quad \text{in } D.$$

Therefore, we obtain that the resolvent equation (9.6) holds true for all $f \in C(\overline{D})$, since the space $C^\theta(\overline{D})$ is dense in $C(\overline{D})$ and since the operators G_α^0 and G_β^0 are bounded.

(d) First, let $f(x)$ be an arbitrary function in $C^\theta(\overline{D})$ satisfying the boundary condition $f|_{\partial D} = 0$. Then it follows from an application of the uniqueness theorem for problem (9.2) (Theorem 9.1) that we have, for all α, β,

$$f - \alpha G_\alpha^0 f = G_\alpha^0 \left((\beta - A) f \right) - \beta G_\alpha^0 f.$$

Indeed, the both sides satisfy the same equation $(\alpha - A)u = -Af$ in D and have the same boundary value 0 on ∂D. Thus we have, by estimate (9.5),

$$\|f - \alpha G_\alpha^0 f\|_\infty \leq \frac{1}{\alpha} \|(\beta - A)f\|_\infty + \frac{\beta}{\alpha} \|f\|_\infty,$$

so that

$$\lim_{\alpha \to +\infty} \|f - \alpha G_\alpha^0 f\|_\infty = 0. \tag{9.10}$$

Now let $f(x)$ be an arbitrary function in $C(\overline{D})$ satisfying the boundary condition $f|_{\partial D} = 0$. By means of mollifiers, we can find a sequence $\{f_j\}$ in $C^\theta(\overline{D})$ such that

$$\begin{cases} f_j \longrightarrow f & \text{in } C(\overline{D}) \text{ as } j \to \infty, \\ f_j = 0 & \text{on } \partial D. \end{cases}$$

Then we have, by estimate (9.5) and assertion (9.10) with $f := f_j$,

$$\begin{aligned} \|f - \alpha G_\alpha^0 f\|_\infty &\leq \|f - f_j\|_\infty + \|f_j - \alpha G_\alpha^0 f_j\|_\infty + \|\alpha G_\alpha^0 (f_j - f)\|_\infty \\ &\leq 2\|f - f_j\|_\infty + \|f_j - \alpha G_\alpha^0 f_j\|_\infty, \end{aligned}$$

and hence

$$\limsup_{\alpha \to +\infty} \|f - \alpha G_\alpha^0 f\|_\infty \leq 2\|f - f_j\|_\infty.$$

This proves the desired assertion (9.8), since $\|f - f_j\|_\infty \to 0$ as $j \to \infty$.

To prove assertion (9.7), let $f(x)$ be an arbitrary function in $C(\overline{D})$ and let x be an arbitrary point of D. If we take a function $\psi(y)$ in $C(\overline{D})$ such that

$$\begin{cases} 0 \leq \psi(y) \leq 1 & \text{on } \overline{D}, \\ \psi(y) = 0 & \text{in a neighborhood of } x, \\ \psi(y) = 1 & \text{near the set } \partial D, \end{cases}$$

then it follows from the non-negativity of G_α^0 and estimate (9.5) that

$$0 \leq \alpha G_\alpha^0 \psi(x) + \alpha G_\alpha^0 (1 - \psi)(x) = \alpha G_\alpha^0 1(x) \leq 1. \tag{9.11}$$

However, by applying assertion (9.8) to the function $1 - \psi(y)$ we have the assertion

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 (1 - \psi)(x) = (1 - \psi)(x) = 1 \quad \text{for all } x \in D.$$

In view of inequalities (9.11), this implies that

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 \psi(x) = 0 \quad \text{for all } x \in D.$$

Thus, since we have the inequalities

$$-\|f\|_\infty \psi \leq f\psi \leq \|f\|_\infty \psi \quad \text{on } \overline{D},$$

it follows that, for $x \in D$,

$$|\alpha G_\alpha^0 (f\psi)(x)| \leq \|f\|_\infty \cdot \alpha G_\alpha^0 \psi(x) \longrightarrow 0 \quad \text{as } \alpha \to +\infty.$$

Therefore, by applying assertion (9.8) to the function $(1 - \psi(y))f(y)$ we obtain that

$$\begin{aligned} f(x) = ((1 - \psi)f)(x) &= \lim_{\alpha \to +\infty} \alpha G_\alpha^0 ((1 - \psi)f)(x) \\ &= \lim_{\alpha \to +\infty} \alpha G_\alpha^0 f(x) \quad \text{for all } x \in D. \end{aligned}$$

This proves the desired assertion (9.7).

(ii) (a') Since the space $C^{2+\theta}(\partial D)$ is dense in $C(\partial D)$, by Lemma 9.3 it follows that the operator $H_\alpha : C^{2+\theta}(\partial D) \to C^{2+\theta}(\overline{D})$ can be uniquely extended to a non-negative, bounded linear operator $H_\alpha : C(\partial D) \to C(\overline{D})$. Furthermore, by applying Theorem A.2 with $A := A - \alpha$ we have the assertion

$$\|H_\alpha\| = \|H_\alpha 1\|_\infty = 1.$$

(b') This assertion follows from formula (9.4), since the space $C^{2+\theta}(\partial D)$ is dense in $C(\partial D)$ and since the operator $H_\alpha : C(\partial D) \to C(\overline{D})$ is bounded.

(c') We find from the uniqueness theorem for problem (9.2) that equation (9.9) holds true for all $\varphi \in C^{2+\theta}(\partial D)$. Indeed, it suffices to note that the function

$$w = H_\alpha \varphi - H_\beta \varphi + (\alpha - \beta) G_\alpha^0 \, H_\beta \varphi \in C^{2+\theta}(\overline{D})$$

satisfies the conditions

$$\begin{cases} (\alpha - A) w = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

so that

$$w = 0 \quad \text{in } D.$$

Therefore, we obtain that the desired equation (9.9) holds true for all $\varphi \in C(\partial D)$, since the space $C^{2+\theta}(\partial D)$ is dense in $C(\partial D)$ and since the operators G_α^0 and H_α are bounded.

The proof of Theorem 9.4 is now complete. □

Summing up, we have the following diagrams for the operators G_α^0 and H_α:

$$\begin{array}{ccc} C(\overline{D}) & \xrightarrow{G_\alpha^0} & C(\overline{D}) \\ \uparrow & & \uparrow \\ C^\theta(\overline{D}) & \xrightarrow{G_\alpha^0} & C^{2+\theta}(\overline{D}) \end{array}$$

$$\begin{array}{ccc} C(\partial D) & \xrightarrow{H_\alpha} & C(\overline{D}) \\ \uparrow & & \uparrow \\ C^{2+\theta}(\partial D) & \xrightarrow{H_\alpha} & C^{2+\theta}(\overline{D}) \end{array}$$

Now we consider the following boundary value problem in the framework of the spaces of *continuous functions*.

$$\begin{cases} (\alpha - A) u = f & \text{in } D, \\ L u = 0 & \text{on } \partial D. \end{cases} \tag{9.12}$$

To do this, we introduce three operators associated with problem (9.12).

(I) First, we introduce a linear operator

$$A : C(\overline{D}) \longrightarrow C(\overline{D})$$

as follows.

(a) The domain $\mathcal{D}(A)$ of A is the space $C^2(\overline{D})$.
(b) $Au = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u$, $u \in \mathcal{D}(A)$.

Then we have the following:

Lemma 9.5. *The operator A has its minimal closed extension $\overline{A}$ in the space $C(\overline{D})$.*

Proof. We apply part (i) of Theorem 2.18 to the operator A.

Assume that a function $u \in C^2(\overline{D})$ takes a positive maximum at an interior point x_0 of D:

$$u(x_0) = \max_{x \in \overline{D}} u(x) > 0.$$

Then it follows that

$$\frac{\partial u}{\partial x_i}(x_0) = 0, \quad 1 \leq i \leq N,$$
$$\sum_{i,j=1}^{N} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \leq 0,$$

since the matrix $(a^{ij}(x))$ is positive definite. Hence we have the assertion

$$Au(x_0) = \sum_{i,j=1}^{N} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) + c(x_0)u(x_0) \leq 0.$$

This implies that the operator A satisfies condition (β) of Theorem 2.18 with $K_0 := D$ and $K := \overline{D}$. Therefore, Lemma 9.5 follows from an application of the same theorem.

The proof of Lemma 9.5 is complete. □

Remark 9.1. Since the injection: $C(\overline{D}) \to \mathcal{D}'(D)$ is continuous, we have the formula

$$\overline{A}u = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u, \quad u \in C(\overline{D}),$$

where the right-hand side is taken in the sense of *distributions*. The operators A and $\overline{A}$ can be visualized as follows:

$$\begin{array}{ccc} C(\overline{D}) & \xrightarrow{\overline{A}} & C(\overline{D}) \\ \uparrow & & \uparrow \\ C^2(\overline{D}) & \xrightarrow{A} & C(\overline{D}) \end{array}$$

The extended operators $G_\alpha^0 : C(\overline{D}) \to C(\overline{D})$ and $H_\alpha : C(\partial D) \to C(\overline{D})$, $\alpha > 0$, still satisfy formulas (9.3) and (9.4) respectively in the following sense:

Lemma 9.6. *(i) For any $f \in C(\overline{D})$, we have the assertions*

$$\begin{cases} G_\alpha^0 f \in \mathcal{D}(\overline{A}), \\ (\alpha I - \overline{A}) G_\alpha^0 f = f \ \textit{in } D. \end{cases}$$

(ii) For any $\varphi \in C(\partial D)$, we have the assertions

$$\begin{cases} H_\alpha \varphi \in \mathcal{D}(\overline{A}), \\ (\alpha I - \overline{A}) H_\alpha \varphi = 0 \ \textit{in } D. \end{cases}$$

Here $\mathcal{D}(\overline{A})$ is the domain of the closed extension $\overline{A}$.

Proof. (i) By making use of Friedrichs' mollifiers, we can choose a sequence $\{f_j\}$ in $C^\theta(\overline{D})$ such that $f_j \to f$ in $C(\overline{D})$ as $j \to \infty$. Then it follows from the boundedness of G_α^0 that

$$G_\alpha^0 f_j \longrightarrow G_\alpha^0 f \quad \text{in } C(\overline{D}),$$

and further that

$$(\alpha - A) G_\alpha^0 f_j = f_j \longrightarrow f \quad \text{in } C(\overline{D}).$$

Hence we have the assertions

$$\begin{cases} G_\alpha^0 f \in \mathcal{D}(\overline{A}), \\ (\alpha I - \overline{A}) G_\alpha^0 f = f \quad \text{in } D. \end{cases}$$

since the operator $\overline{A} : C(\overline{D}) \to C(\overline{D})$ is closed.

(ii) Similarly, part (ii) is proved, since the space $C^{2+\theta}(\partial D)$ is dense in $C(\partial D)$ and since the operator $H_\alpha : C(\partial D) \to C(\overline{D})$ is bounded.

The proof of Lemma 9.6 is complete. □

Corollary 9.7. *Every function u in $\mathcal{D}(\overline{A})$ can be written in the following form:*

$$u = G_\alpha^0 \left((\alpha I - \overline{A}) u \right) + H_\alpha (u|_{\partial D}) \quad \textit{for all } \alpha > 0. \tag{9.13}$$

Proof. We let

$$w = u - G_\alpha^0 \left((\alpha I - \overline{A}) u \right) - H_\alpha (u|_{\partial D}).$$

Then it follows from Lemma 9.6 that the function w is in $\mathcal{D}(\overline{A})$ and satisfies the conditions

$$\begin{cases} (\alpha I - \overline{A}) w = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

However, in light of Remark 9.1, by applying Lemma 5.1 with $\lambda_0 := \alpha$ and Theorem 4.9 with $A := A - \alpha$ to the Dirichlet case ($\mu(x') \equiv 0$ and $\gamma(x') \equiv -1$ on ∂D) we obtain that

$$w \in C^\infty(\overline{D}).$$

Therefore, it follows from an application of Theorem 9.1 with $A := A - \alpha$ that

$$w = 0.$$

This proves the desired formula (9.13).

The proof of Corollary 9.7 is complete. □

(II) Secondly, we introduce a linear operator

$$LG_\alpha^0 : C(\overline{D}) \longrightarrow C(\partial D)$$

as follows.

(a) The domain $\mathcal{D}\left(LG_\alpha^0\right)$ of LG_α^0 is the Hölder space $C^\theta(\overline{D})$ with $0 < \theta < 1$.
(b) $LG_\alpha^0 f = L\left(G_\alpha^0 f\right)$, $f \in \mathcal{D}\left(LG_\alpha^0\right)$.

Then we have the following:

Lemma 9.8. *The operator LG_α^0, $\alpha > 0$, can be uniquely extended to a non-negative, bounded linear operator $\overline{LG_\alpha^0} : C(\overline{D}) \to C(\partial D)$.*

Proof. Let $f(x)$ be an arbitrary function in $\mathcal{D}(LG_\alpha^0) = C^\theta(\overline{D})$ such that $f(x) \geq 0$ on $\overline{D}$. Then we have the assertions

$$\begin{cases} G_\alpha^0 f \in C^{2+\theta}(\overline{D}), & \\ G_\alpha^0 f \geq 0 & \text{on } \overline{D}, \\ G_\alpha^0 f|_{\partial D} = 0 & \text{on } \partial D, \end{cases}$$

and hence

$$\begin{aligned} LG_\alpha^0 f &= \mu(x') \frac{\partial}{\partial \mathbf{n}} (G_\alpha^0 f) + \gamma(x') G_\alpha^0 f \\ &= \mu(x') \frac{\partial}{\partial \mathbf{n}} (G_\alpha^0 f) \geq 0 \quad \text{on } \partial D. \end{aligned}$$

This proves that the operator LG_α^0 is non-negative.

By the non-negativity of LG_α^0, we have, for all $f \in \mathcal{D}(LG_\alpha^0)$,

$$-LG_\alpha^0 \|f\|_\infty \leq LG_\alpha^0 f \leq LG_\alpha^0 \|f\|_\infty \quad \text{on } \partial D.$$

This implies the boundedness of LG_α^0 with norm

$$\|LG_\alpha^0\| = \|LG_\alpha^0 1\|_\infty.$$

Recall that the space $C^\theta(\overline{D})$ is dense in $C(\overline{D})$ and that non-negative functions can be approximated by non-negative smooth functions. Hence we find that the operator LG_α^0 can be uniquely extended to a non-negative, bounded linear operator $\overline{LG_\alpha^0} : C(\overline{D}) \to C(\partial D)$.

The proof of Lemma 9.8 is complete. □

The operators LG_α^0 and $\overline{LG_\alpha^0}$ can be visualized as follows:

$$\begin{array}{ccc} C(\overline{D}) & \xrightarrow{\overline{LG_\alpha^0}} & C(\partial D) \\ \uparrow & & \uparrow \\ C^\theta(\overline{D}) & \xrightarrow{LG_\alpha^0} & C^{1+\theta}(\partial D) \end{array}$$

The next lemma states a fundamental relationship between the operators $\overline{LG_\alpha^0}$ and $\overline{LG_\beta^0}$ for α, $\beta > 0$:

Lemma 9.9. *For any $f \in C(\overline{D})$, we have the formula*

$$\overline{LG_\alpha^0} f - \overline{LG_\beta^0} f + (\alpha - \beta)\overline{LG_\alpha^0}\, G_\beta^0 f = 0 \quad \text{for all } \alpha,\ \beta > 0. \tag{9.14}$$

Proof. Choose a sequence $\{f_j\}$ in $C^\theta(\overline{D})$ such that $f_j \to f$ in $C(\overline{D})$ as $j \to \infty$, just as in Lemma 9.6. Then, by using the resolvent equation (9.6) with $f := f_j$ we have the formula

$$LG_\alpha^0 f_j - LG_\beta^0 f_j + (\alpha - \beta) LG_\alpha^0 G_\beta^0 f_j = 0.$$

Hence, the desired formula (9.14) follows by letting $j \to \infty$, since the operators $\overline{LG_\alpha^0}$, $\overline{LG_\beta^0}$ and G_β^0 are all bounded.

The proof of Lemma 9.9 is complete. □

(III) Finally, we introduce a linear operator

$$LH_\alpha : C(\partial D) \longrightarrow C(\partial D)$$

as follows.

(a) The domain $\mathcal{D}(LH_\alpha)$ of LH_α is the space $C^{2+\theta}(\partial D)$.
(b) $LH_\alpha \psi = L(H_\alpha \psi)$, $\psi \in \mathcal{D}(LH_\alpha)$.

Then we have the following:

Lemma 9.10. *The operator LH_α, $\alpha > 0$, has its minimal closed extension $\overline{LH_\alpha}$ in the space $C(\partial D)$.*

Proof. We apply part (i) of Theorem 2.18 to the operator LH_α. To do this, it suffices to show that the operator LH_α satisfies condition (β') with $K := \partial D$ (or condition (β) with $K := K_0 = \partial D$) of the same theorem.

Assume that a function ψ in $\mathcal{D}(LH_\alpha) = C^{2+\theta}(\partial D)$ takes its positive maximum at some point x' of ∂D. Since the function $H_\alpha\psi$ is in $C^{2+\theta}(\overline{D})$ and satisfies

$$\begin{cases} (A - \alpha) H_\alpha \psi = 0 & \text{in } D, \\ H_\alpha \psi = \psi & \text{on } \partial D, \end{cases}$$

by applying the weak maximum principle (Theorem A.1) with $A := A - \alpha$ to the function $H_\alpha\psi$ we find that the function $H_\alpha\psi$ takes its positive maximum at the boundary point $x' \in \partial D$. Thus we can apply the boundary point lemma (Lemma A.3) with $A := A - \alpha$ to obtain that

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha \psi)(x') < 0.$$

Hence we have the inequality

$$\begin{aligned} LH_\alpha \psi(x') &= \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x') + \mu(x') \frac{\partial}{\partial \mathbf{n}}(H_\alpha \psi)(x') \\ &\quad + \gamma(x')\psi(x') \\ &\leq 0. \end{aligned}$$

This verifies condition (β') of Theorem 2.18. Therefore, Lemma 9.10 follows from an application of the same theorem.

The proof of Lemma 9.10 is complete. □

Remark 9.2. The operator $\overline{LH_\alpha}$ enjoys the following property:

> If a function ψ in the domain $\mathcal{D}\left(\overline{LH_\alpha}\right)$ takes its *positive maximum* at some point x' of ∂D, then we have the inequality
> $$\overline{LH_\alpha}\psi(x') \leq 0. \tag{9.15}$$

The operators LH_α and $\overline{LH_\alpha}$ can be visualized as follows:

$$\begin{array}{ccc} C(\partial D) & \xrightarrow{\overline{LH_\alpha}} & C(\partial D) \\ \uparrow & & \uparrow \\ C^{2+\theta}(\partial D) & \xrightarrow{LH_\alpha} & C^{1+\theta}(\partial D) \end{array}$$

The next lemma states a fundamental relationship between the operators $\overline{LH_\alpha}$ and $\overline{LH_\beta}$ for α, $\beta > 0$:

Lemma 9.11. *The domain $\mathcal{D}\left(\overline{LH_\alpha}\right)$ of $\overline{LH_\alpha}$ does not depend on $\alpha > 0$; so we denote by $\mathcal{D}$ the common domain. Then we have the formula*

$$\overline{LH_\alpha}\psi - \overline{LH_\beta}\psi + (\alpha - \beta)\overline{LG_\alpha^0}\, H_\beta \psi = 0 \quad \text{for all } \alpha,\ \beta > 0 \text{ and } \psi \in \mathcal{D}. \tag{9.16}$$

Proof. Let $\psi(x')$ be an arbitrary function in $\mathcal{D}(\overline{LH_\beta})$, and choose a sequence $\{\psi_j\}$ in $\mathcal{D}(LH_\beta) = C^{2+\theta}(\partial D)$ such that, as $j \to \infty$,

$$\begin{cases} \psi_j \longrightarrow \psi & \text{in } C(\partial D), \\ LH_\beta \psi_j \longrightarrow \overline{LH_\beta}\psi & \text{in } C(\partial D). \end{cases}$$

Then it follows from the boundedness of H_β and $\overline{LG_\alpha^0}$ that

$$LG_\alpha^0(H_\beta\psi_j) = \overline{LG_\alpha^0}(H_\beta\psi_j) \longrightarrow \overline{LG_\alpha^0}(H_\beta\psi) \quad \text{in } C(\partial D).$$

Therefore, by using formula (9.9) with $\varphi := \psi_j$ we obtain that, as $j \to \infty$,

$$\begin{aligned} LH_\alpha\psi_j &= LH_\beta\psi_j - (\alpha - \beta)LG_\alpha^0(H_\beta\psi_j) \\ &\longrightarrow \overline{LH_\beta}\psi - (\alpha - \beta)\overline{LG_\alpha^0}(H_\beta\psi) \quad \text{in } C(\partial D). \end{aligned}$$

This implies that

$$\begin{cases} \psi \in \mathcal{D}(\overline{LH_\alpha}), \\ \overline{LH_\alpha}\psi = \overline{LH_\beta}\psi - (\alpha - \beta)\overline{LG_\alpha^0}(H_\beta\psi), \end{cases}$$

since the operator $\overline{LH_\alpha} : C(\partial D) \to C(\partial D)$ is closed.

Conversely, by interchanging α and β we have the assertion

$$\mathcal{D}(\overline{LH_\alpha}) \subset \mathcal{D}(\overline{LH_\beta}),$$

and so

$$\mathcal{D}(\overline{LH_\alpha}) = \mathcal{D}(\overline{LH_\beta}).$$

The proof of Lemma 9.11 is complete. □

Now we can prove a general existence theorem for Feller semigroups on ∂D in terms of boundary value problem (9.12). The next theorem asserts that the operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on ∂D if and only if problem (9.12) is solvable for *sufficiently many* functions φ in the space $C(\partial D)$:

Theorem 9.12. *(i) If the operator $\overline{LH_\alpha}$, $\alpha > 0$, is the infinitesimal generator of a Feller semigroup on ∂D, then, for each positive constant λ, the boundary value problem*

$$\begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \partial D \end{cases} \tag{9.17}$$

has a solution $u \in C^{2+\theta}(\overline{D})$ for any φ in some dense *subset of $C(\partial D)$.*

(ii) Conversely, if, for some non-negative constant λ, problem (9.17) has a solution $u \in C^{2+\theta}(\overline{D})$ for any φ in some dense subset of $C(\partial D)$, then the operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on ∂D.

Proof. (i) If the operator $\overline{LH_\alpha}$ generates a Feller semigroup on ∂D, by applying part (i) of Theorem 2.18 with $K := \partial D$ to the operator $\overline{LH_\alpha}$ we obtain that

$$\mathcal{R}\left(\lambda I - \overline{LH_\alpha}\right) = C(\partial D) \quad \text{for each } \lambda > 0.$$

This implies that the range $\mathcal{R}\left(\lambda I - LH_\alpha\right)$ is a dense subset of $C(\partial D)$ for each $\lambda > 0$. However, if $\varphi \in C(\partial D)$ is in the range $\mathcal{R}\left(\lambda I - LH_\alpha\right)$ and if $\varphi = \left(\lambda I - LH_\alpha\right)\psi$ with $\psi \in C^{2+\theta}(\partial D)$, then the function $u = H_\alpha\psi \in C^{2+\theta}(\overline{D})$ is a solution of problem (9.17). This proves part (i) of Theorem 9.12.

(ii) We apply part (ii) of Theorem 2.18 with $K := \partial D$ to the operator LH_α. To do this, it suffices to show that the operator LH_α satisfies condition (γ) of the same theorem, since it satisfies condition (β'), as is shown in the proof of Lemma 9.10.

By the uniqueness theorem for problem (9.2), it follows that any function $u \in C^{2+\theta}(\overline{D})$ which satisfies the homogeneous equation

$$(\alpha - A)u = 0 \quad \text{in } D$$

can be written in the form:

$$u = H_\alpha\left(u|_{\partial D}\right), \quad u|_{\partial D} \in C^{2+\theta}(\partial D) = \mathcal{D}\left(LH_\alpha\right).$$

Thus we find that if there exists a solution $u \in C^{2+\theta}(\overline{D})$ of problem (9.17) for a function $\varphi \in C(\partial D)$, then we have the assertion

$$\left(\lambda I - LH_\alpha\right)\left(u|_{\partial D}\right) = \varphi,$$

and so

$$\varphi \in \mathcal{R}\left(\lambda I - LH_\alpha\right).$$

Hence, if, for some non-negative constant λ, problem (9.12) has a solution $u \in C^{2+\theta}(\overline{D})$ for any φ in some dense subset of $C(\partial D)$, then the range $\mathcal{R}\left(\lambda I - LH_\alpha\right)$ is dense in $C(\partial D)$. This verifies condition (γ) (with $\alpha_0 := \lambda$) of Theorem 2.18. Therefore, part (ii) of Theorem 9.12 follows from an application of the same theorem.

Now the proof of Theorem 9.12 is complete. □

Remark 9.3. Intuitively, Theorem 9.12 asserts that we can "piece together" a Markov process (Feller semigroup) on the boundary ∂D with A-diffusion in the interior D to construct a Markov process (Feller semigroup) on the closure $\overline{D} = D \cup \partial D$. The situation may be represented schematically by Figure 9.1.

We conclude this section by giving a precise meaning to the boundary conditions Lu for functions u in $\mathcal{D}(\overline{A})$.

We let

$$\mathcal{D}(L) = \left\{u \in \mathcal{D}(\overline{A}) : u|_{\partial D} \in \mathcal{D}\right\},$$

where $\mathcal{D}$ is the common domain of the operators $\overline{LH_\alpha}$ for all $\alpha > 0$. We remark that the space $\mathcal{D}(L)$ contains $C^{2+\theta}(\overline{D})$, since $C^{2+\theta}(\partial D) = \mathcal{D}\left(LH_\alpha\right) \subset \mathcal{D}$. Corollary 9.7 asserts that every function u in $\mathcal{D}(L) \subset \mathcal{D}(\overline{A})$ can be written in the form

$$u = G_\alpha^0\left((\alpha I - \overline{A})u\right) + H_\alpha\left(u|_{\partial D}\right) \quad \text{for all } \alpha > 0. \tag{9.13}$$

Then we define the boundary condition Lu by the formula

$$Lu = \overline{LG_\alpha^0}\left((\alpha I - \overline{A})u\right) + \overline{LH_\alpha}\left(u|_{\partial D}\right), \quad u \in \mathcal{D}(L). \tag{9.18}$$

The next lemma justifies definition (9.18) of Lu for each $u \in \mathcal{D}(L)$:

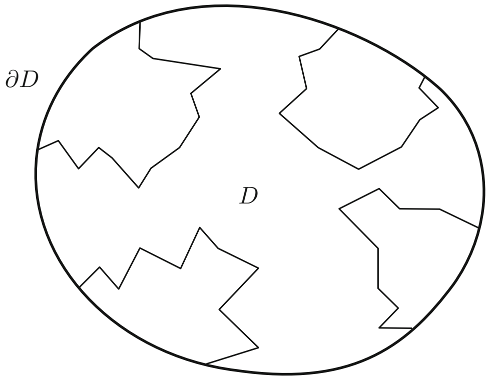

Fig. 9.1.

Lemma 9.13. *The right-hand side of formula (9.18) depends only on u, not on the choice of expression (9.13).*

Proof. Assume that

$$\begin{aligned} u &= G_\alpha^0 \left(\left(\alpha I - \overline{A} \right) u \right) + H_\alpha \left(u|_{\partial D} \right) \\ &= G_\beta^0 \left(\left(\beta I - \overline{A} \right) u \right) + H_\beta \left(u|_{\partial D} \right), \end{aligned}$$

where α, $\beta > 0$. Then it follows from formula (9.14) with $f := \left(\alpha I - \overline{A} \right) u$ and formula (9.18) with $\psi := u|_{\partial D}$ that

$$\begin{aligned} &\overline{LG_\alpha^0} \left(\left(\alpha I - \overline{A} \right) u \right) + \overline{LH_\alpha} \left(u|_{\partial D} \right) \\ &= \overline{LG_\beta^0} \left(\left(\alpha I - \overline{A} \right) u \right) - (\alpha - \beta) \overline{LG_\alpha^0} G_\beta^0 \left(\left(\alpha I - \overline{A} \right) u \right) \\ &\quad + \overline{LH_\beta} \left(u|_{\partial D} \right) - (\alpha - \beta) \overline{LG_\alpha^0} H_\beta \left(u|_{\partial D} \right) \\ &= \overline{LG_\beta^0} \left((\beta I - A) u \right) + \overline{LH_\beta} \left(u|_{\partial D} \right) \\ &\quad + (\alpha - \beta) \left\{ \overline{LG_\beta^0} u - \overline{LG_\alpha^0} G_\beta^0 \left(\alpha I - \overline{A} \right) u - \overline{LG_\alpha^0} H_\beta \left(u|_{\partial D} \right) \right\}. \quad (9.19) \end{aligned}$$

However, the last term of formula (9.19) vanishes. Indeed, it follows from formula (9.13) with $\alpha := \beta$ and formula (9.14) with $f := u$ that

$$\begin{aligned} &\overline{LG_\beta^0} u - \overline{LG_\alpha^0} \left(G_\beta^0 \left(\alpha I - \overline{A} \right) u \right) - \overline{LG_\alpha^0} H_\beta \left(u|_{\partial D} \right) \\ &= \overline{LG_\beta^0} u - \overline{LG_\alpha^0} \left(G_\beta^0 \left(\beta I - \overline{A} \right) u + H_\beta \left(u|_{\partial D} \right) + (\alpha - \beta) G_\beta^0 u \right) \\ &= \overline{LG_\beta^0} u - \overline{LG_\alpha^0} u - (\alpha - \beta) \overline{LG_\alpha^0} G_\beta^0 u \\ &= 0. \end{aligned}$$

Therefore, we obtain from formula (9.19) that

$$\overline{LG_\alpha^0} \left(\left(\alpha I - \overline{A} \right) u \right) + \overline{LH_\alpha} \left(u|_{\partial D} \right) = \overline{LG_\beta^0} \left(\left(\beta I - \overline{A} \right) u \right) + \overline{LH_\beta} \left(u|_{\partial D} \right).$$

The proof of Lemma 9.13 is complete. □

9.2 Feller Semigroups with Reflecting Barrier

Now we consider the Neumann boundary condition

$$L_N u \equiv \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\partial D}.$$

We recall that the boundary condition L_N is supposed to correspond to the *reflection phenomenon.*

The next theorem (formula (9.21)) asserts that we can "piece together" a Markov process on ∂D with A-diffusion in D to construct a Markov process on $\overline{D} = D \cup \partial D$ with reflecting barrier (cf. [BCP, Théorème XIX]):

Theorem 9.14. *We define a linear operator*

$$\mathfrak{A}_N : C(\overline{D}) \longrightarrow C(\overline{D})$$

as follows.

(a) The domain $\mathcal{D}(\mathfrak{A}_N)$ of $\mathfrak{A}_N$ is the space

$$\mathcal{D}(\mathfrak{A}_N) = \left\{u \in \mathcal{D}(\overline{A}) : u|_{\partial D} \in \mathcal{D}_N,\ L_N u = 0\right\}, \tag{9.20}$$

where $\mathcal{D}_N$ is the common domain of the operators $\overline{L_N H_\alpha}$ for all $\alpha > 0$.
(b) $\mathfrak{A}_N u = \overline{A}u$, $u \in \mathcal{D}(\mathfrak{A}_N)$.

Then the operator $\mathfrak{A}_N$ is the infinitesimal generator of some Feller semigroup on $\overline{D}$, and the Green operator $G_\alpha^N = (\alpha I - \mathfrak{A}_N)^{-1}$, $\alpha > 0$, is given by the following formula:

$$G_\alpha^N f = G_\alpha^0 f - H_\alpha \left(\overline{L_N H_\alpha}^{-1} \left(\overline{L_N G_\alpha^0 f}\right)\right), \quad f \in C(\overline{D}). \tag{9.21}$$

Proof. We apply part (ii) of Theorem 2.16 to the operator $\mathfrak{A}_N$ defined by formula (9.20). The proof is divided into eight steps.

Step 1: First, we prove that:

The operator $\overline{L_N H_\alpha}$ is the generator of some Feller semigroup on ∂D, for any *sufficiently large* $\alpha > 0$.

We introduce a linear operator

$$\mathcal{T}_N(\alpha) : B^{2-1/p,p}(\partial D) \longrightarrow B^{2-1/p,p}(\partial D)$$

as follows.

(a) The domain $\mathcal{D}(\mathcal{T}_N(\alpha))$ of $\mathcal{T}_N(\alpha)$ is the space

$$\mathcal{D}(\mathcal{T}_N(\alpha)) = \left\{\varphi \in B^{2-1/p,p}(\partial D) : L_N H_\alpha \varphi \in B^{2-1/p,p}(\partial D)\right\}.$$

(b) $\mathcal{T}_N(\alpha)\varphi = L_N H_\alpha \varphi$, $\varphi \in \mathcal{D}(\mathcal{T}_N(\alpha))$.

Here it should be emphasized that the harmonic operator H_α is essentially the same as the Poisson operator $P(\alpha)$ introduced in Chapter 5.

Then, by arguing just as in the proof of Theorem 7.1 with $\mu(x') := 1$ and $\gamma(x') := 0$ on ∂D we obtain that

$$\begin{aligned}&\text{The operator } \mathcal{T}_N(\alpha) \text{ is } \textit{bijective} \text{ for any sufficiently large } \alpha > 0.\\ &\text{Furthermore, it maps the space } C^\infty(\partial D) \text{ onto itself.}\end{aligned} \tag{9.22}$$

Since we have the assertion

$$L_N H_\alpha = \mathcal{T}_N(\alpha) \quad \text{on } C^\infty(\partial D),$$

it follows from assertion (9.22) that the operator $L_N H_\alpha$ also maps $C^\infty(\partial D)$ onto itself, for any sufficiently large $\alpha > 0$. This implies that the range $\mathcal{R}(L_N H_\alpha)$ is a *dense* subset of $C(\partial D)$. Hence, by applying part (ii) of Theorem 9.12 we obtain that the operator $\overline{L_N H_\alpha}$ generates a Feller semigroup on ∂D, for any sufficiently large $\alpha > 0$.

Step 2: Next we prove that:

$$\begin{aligned}&\text{The operator } \overline{L_N H_\beta} \text{ generates a Feller semigroup on } \partial D,\\ &\text{for } \textit{any } \beta > 0.\end{aligned}$$

We take a positive constant α so large that the operator $\overline{L_N H_\alpha}$ generates a Feller semigroup on ∂D. We apply Corollary 2.19 with $K := \partial D$ to the operator $\overline{L_N H_\beta}$ for $\beta > 0$. By formula (9.16), it follows that the operator $\overline{L_N H_\beta}$ can be written as

$$\overline{L_N H_\beta} = \overline{L_N H_\alpha} + N_{\alpha\beta},$$

where $N_{\alpha\beta} = (\alpha - \beta)\overline{L_N G_\alpha^0}\, H_\beta$ is a bounded linear operator on $C(\partial D)$ into itself. Furthermore, assertion (9.16) implies that the operator $\overline{L_N H_\beta}$ satisfies condition (β') of Theorem 2.18. Therefore, it follows from an application of Corollary 2.19 that the operator $\overline{L_N H_\beta}$ also generates a Feller semigroup on ∂D.

Step 3: Now we prove that:

The equation

$$\overline{L_N H_\alpha}\, \psi = \varphi$$

has a unique solution ψ in $\mathcal{D}(\overline{L_N H_\alpha})$ for any $\varphi \in C(\partial D)$; hence the inverse $\overline{L_N H_\alpha}^{-1}$ of $\overline{L_N H_\alpha}$ can be defined on the whole space $C(\partial D)$. Furthermore, the operator $-\overline{L_N H_\alpha}^{-1}$ is non-negative and bounded on the space $C(\partial D)$. (9.23)

Since the function $H_\alpha 1(x)$ takes its positive maximum 1 only on the boundary ∂D, we can apply the boundary point lemma (Lemma A.3) to obtain that

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) < 0 \quad \text{on } \partial D. \tag{9.24}$$

Hence the Neumann boundary condition implies that

$$L_N H_\alpha 1 = \frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) < 0 \quad \text{on } \partial D,$$

and so

$$\ell_\alpha = \min_{x' \in \partial D} (-L_N H_\alpha 1)(x') = -\max_{x' \in \partial D} L_N H_\alpha 1(x') > 0.$$

Furthermore, by using Corollary 2.17 with $K := \partial D$, $A := \overline{L_N H_\alpha}$ and $c := \ell_\alpha$ we obtain that the operator $\overline{L_N H_\alpha} + \ell_\alpha I$ is the infinitesimal generator of some Feller semigroup on ∂D. Therefore, since $\ell_\alpha > 0$, it follows from an application of part (i) of Theorem 2.16 with $\mathfrak{A} := \overline{L_N H_\alpha} + \ell_\alpha I$ that the equation

$$-\overline{L_N H_\alpha}\, \psi = \left(\ell_\alpha I - (\overline{L_N H_\alpha} + \ell_\alpha I)\right) \psi = \varphi$$

has a unique solution $\psi \in \mathcal{D}(\overline{L_N H_\alpha})$ for any $\varphi \in C(\partial D)$, and further that the operator $-\overline{L_N H_\alpha}^{-1} = \left(\ell_\alpha I - (\overline{L_N H_\alpha} + \ell_\alpha I)\right)^{-1}$ is non-negative and bounded on the space $C(\partial D)$ with norm

$$\left\| -\overline{L_N H_\alpha}^{-1} \right\| = \left\| \left(\ell_\alpha I - (\overline{L_N H_\alpha} + \ell_\alpha I)\right)^{-1} \right\| \leq \frac{1}{\ell_\alpha}.$$

Step 4: By assertion (9.23), we can define the right-hand side of formula (9.21) for all $\alpha > 0$. We prove that:

$$G_\alpha^N = (\alpha I - \mathfrak{A}_N)^{-1}, \quad \alpha > 0. \tag{9.25}$$

In view of Lemmas 9.6 and 9.13 with $L := L_N$, it follows that we have, for all $f \in C(\overline{D})$,

$$\begin{cases} G_\alpha^N f = G_\alpha^0 f - H_\alpha \left(\overline{L_N H_\alpha}^{-1} \left(\overline{L_N G_\alpha^0} f \right) \right) \in \mathcal{D}(\overline{A}), \\ G_\alpha^N f|_{\partial D} = -\overline{L_N H_\alpha}^{-1} \left(\overline{L_N G_\alpha^0} f \right) \in \mathcal{D}\left(\overline{L_N H_\alpha} \right) = \mathcal{D}_N, \\ L_N (G_\alpha^N f) = \overline{L_N G_\alpha^0} f - \overline{L_N H_\alpha} \left(\overline{L_N H_\alpha}^{-1} \left(\overline{L_N G_\alpha^0} f \right) \right) = 0, \end{cases}$$

and

$$(\alpha I - \overline{A})(G_\alpha^N f) = f.$$

Hence we have proved that, for all $f \in C(\overline{D})$,

$$\begin{cases} G_\alpha^N f \in \mathcal{D}(\mathfrak{A}_N), \\ (\alpha I - \mathfrak{A}_N) G_\alpha^N f = f. \end{cases}$$

This proves that
$$(\alpha I - \mathfrak{A}_N)\, G_\alpha^N = I \quad \text{on } C(\overline{D}).$$

Therefore, in order to prove formula (9.25) it suffices to show the injectivity of the operator $\alpha I - \mathfrak{A}_N$ for $\alpha > 0$.

Assume that:
$$u \in \mathcal{D}\,(\mathfrak{A}_N) \ \text{ and } \ (\alpha I - \mathfrak{A}_N)\, u = 0.$$

Then, by Corollary 9.7 it follows that the function u can be written as
$$u = H_\alpha\,(u|_{\partial D})\,, \ u|_{\partial D} \in \mathcal{D}_N = \mathcal{D}\left(\overline{L_N H_\alpha}\right).$$

Thus we have the assertion
$$\overline{L_N H_\alpha}\,(u|_{\partial D}) = L_N u = 0.$$

In view of assertion (9.23), this implies that
$$u|_{\partial D} = 0,$$

so that
$$u = H_\alpha\,(u|_{\partial D}) = 0 \quad \text{in } D.$$

This proves the injectivity of $\alpha I - \mathfrak{A}_N$ for $\alpha > 0$.

Step 5: The non-negativity of G_α^N, $\alpha > 0$, follows immediately from formula (9.21), since the operators G_α^0, H_α, $-\overline{L_N H_\alpha}^{-1}$ and $\overline{L_N G_\alpha^0}$ are all non-negative.

Step 6: We prove that the operator G_α^N is bounded on the space $C(\overline{D})$ with norm
$$\left\|G_\alpha^N\right\| \leq \frac{1}{\alpha} \quad \text{for all } \alpha > 0. \tag{9.26}$$

To do this, it suffices to show that, for all $\alpha > 0$,
$$G_\alpha^N 1 \leq \frac{1}{\alpha} \quad \text{on } \overline{D}. \tag{9.27}$$

since G_α^N is non-negative on $C(\overline{D})$.

First, it follows from the uniqueness property of solutions of problem (9.2) that
$$\alpha G_\alpha^0 1 + H_\alpha 1 = 1 + G_\alpha^0(c(x)) \quad \text{on } \overline{D}. \tag{9.28}$$

Indeed, the both sides satisfy the same equation $(\alpha - A)u = \alpha$ in D and have the same boundary value 1 on ∂D.

By applying the boundary operator L_N to the both hand sides of equality (9.28), we obtain that
$$\begin{aligned}
-L_N H_\alpha 1 &= -L_N 1 - L_N G_\alpha^0(c(x)) + \alpha L_N G_\alpha^0 1 \\
&= -\frac{\partial}{\partial \mathbf{n}}\left(G_\alpha^0(c(x))\right)\Big|_{\partial D} + \alpha L_N G_\alpha^0 1 \\
&\geq \alpha L_N G_\alpha^0 1 \quad \text{on } \partial D,
\end{aligned}$$

since $G_\alpha^0(c(x)) = 0$ on ∂D and $G_\alpha^0(c(x)) \leq 0$ on $\overline{D}$. Hence we have, by the non-negativity of $-\overline{L_N H_\alpha}^{-1}$,

$$-\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 1\right) \leq \frac{1}{\alpha} \quad \text{on } \partial D. \tag{9.29}$$

By using formula (9.21) with $f := 1$, inequality (9.29) and equality (9.28), we obtain that

$$\begin{aligned} G_\alpha^N 1 &= G_\alpha^0 1 + H_\alpha\left(-\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 1\right)\right) \\ &\leq G_\alpha^0 1 + \frac{1}{\alpha} H_\alpha 1 \\ &= \frac{1}{\alpha} + \frac{1}{\alpha} G_\alpha^0(c(x)) \\ &\leq \frac{1}{\alpha} \quad \text{on } \overline{D}, \end{aligned}$$

since the operators H_α and G_α^0 are non-negative and since $G_\alpha^0(c(x)) \leq 0$ on $\overline{D}$.

Therefore, we have proved the desired assertion (9.27) for all $\alpha > 0$.

Step 7: Finally, we prove that:

$$\text{The domain } \mathcal{D}\left(\mathfrak{A}_N\right) \text{ is } \textit{dense} \text{ in the space } C(\overline{D}). \tag{9.30}$$

Step 7-1: Before the proof, we need some lemmas on the behavior of G_α^0, H_α and $-\overline{L_N H_\alpha}^{-1}$ as $\alpha \to +\infty$:

Lemma 9.15. *For all $f \in C(\overline{D})$, we have the assertion*

$$\lim_{\alpha \to +\infty}\left[\alpha G_\alpha^0 f + H_\alpha\left(f|_{\partial D}\right)\right] = f \quad \textit{in } C(\overline{D}). \tag{9.31}$$

Proof. Choose a positive constant β and let

$$g := f - H_\beta(f|_{\partial D}).$$

Then, by using formula (9.9) with $\varphi := f|_{\partial D}$ we obtain that

$$\alpha G_\alpha^0 g - g = \left[\alpha G_\alpha^0 f + H_\alpha(f|_{\partial D}) - f\right] - \beta G_\alpha^0 H_\beta(f|_{\partial D}). \tag{9.32}$$

However, we have, by estimate (9.5),

$$\lim_{\alpha \to +\infty} G_\alpha^0 H_\beta(f|_{\partial D}) = 0 \quad \text{in } C(\overline{D}),$$

and, by assertion (9.8),

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 g = g \quad \text{in } C(\overline{D}),$$

since $g|_{\partial D} = 0$.

Therefore, the desired formula (9.31) follows by letting $\alpha \to +\infty$ in formula (9.32).

The proof of Lemma 9.15 is complete. $\square$

Lemma 9.16. *The function*

$$\frac{\partial}{\partial \mathbf{n}} (H_\alpha 1)\bigg|_{\partial D}$$

diverges to $-\infty$ *uniformly and monotonically as* $\alpha \to +\infty$.

Proof. First, formula (9.9) with $\varphi := 1$ gives that

$$H_\alpha 1 = H_\beta 1 - (\alpha - \beta) G_\alpha^0 H_\beta 1.$$

Thus, in view of the non-negativity of G_α^0 and H_α it follows that

$$\alpha \geq \beta \implies H_\alpha 1 \leq H_\beta 1 \quad \text{on } \overline{D}.$$

Since $H_\alpha 1|_{\partial D} = H_\beta 1|_{\partial D} = 1$, this implies that the functions

$$\frac{\partial}{\partial \mathbf{n}} (H_\alpha 1)\bigg|_{\partial D}$$

are monotonically non-increasing in α. Furthermore, by using formula (9.7) with $f := H_\beta 1$ we find that the function

$$H_\alpha 1(x) = H_\beta 1(x) - \left(1 - \frac{\beta}{\alpha}\right) \alpha G_\alpha^0 H_\beta 1(x)$$

converges to zero monotonically as $\alpha \to +\infty$, for each interior point x of D.

Now, for any given positive constant K we can construct a function $u \in C^2(\overline{D})$ such that

$$u = 1 \quad \text{on } \partial D, \tag{9.33a}$$

$$\frac{\partial u}{\partial \mathbf{n}} \leq -K \quad \text{on } \partial D. \tag{9.33b}$$

Indeed, it follows from an application of Theorem 9.1 that, for any integer $m > 0$, the function

$$u = (H_{\alpha_0} 1)^m$$

belongs to $C^{2+\theta}(\overline{D})$ and satisfies condition (9.33a). Furthermore, we have the assertion

$$\begin{aligned} \frac{\partial u}{\partial \mathbf{n}}\bigg|_{\partial D} &= m \, \frac{\partial}{\partial \mathbf{n}} (H_{\alpha_0} 1)\bigg|_{\partial D} \\ &\leq m \max_{x' \in \partial D} \left(\frac{\partial}{\partial \mathbf{n}} (H_{\alpha_0} 1)(x') \right). \end{aligned}$$

In view of inequality (9.24) with $\alpha := \alpha_0$, this implies that the function $u = (H_{\alpha_0} u)^m$ satisfies condition (9.33b) for m sufficiently large.

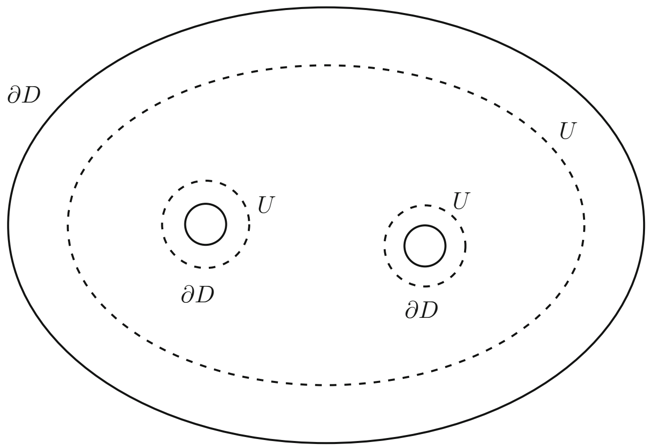

Fig. 9.2.

We take a function $u \in C^2(\overline{D})$ which satisfies conditions (9.33a) and (9.33b), and choose a neighborhood U of ∂D, relative to $\overline{D}$, with smooth boundary ∂U such that (see Figure 9.2)

$$u \geq \frac{1}{2} \quad \text{on } U. \tag{9.34}$$

Recall that the function $H_\alpha 1$ converges to zero in the interior D monotonically as $\alpha \to +\infty$. Since $u = H_\alpha 1 = 1$ on the boundary ∂D, by using Dini's theorem we can find a positive constant α (depending on u and hence on K) such that

$$H_\alpha 1 \leq u \quad \text{on } \partial U \setminus \partial D, \tag{9.35a}$$

$$\alpha > 2\|Au\|_\infty. \tag{9.35b}$$

It follows from inequalities (9.34) and (9.35b) that

$$\begin{aligned} (A-\alpha)(H_\alpha 1 - u) &= \alpha u - Au \\ &\geq \frac{\alpha}{2} - \|Au\|_\infty \\ &> 0 \quad \text{in } U. \end{aligned}$$

Thus, by applying the weak maximum principle (Theorem A.1) with $A := A - \alpha$ to the function $H_\alpha 1 - u$ we obtain that the function $H_\alpha 1 - u$ may take its positive maximum only on the boundary ∂U. However, conditions (9.33a) and (9.35a) imply that

$$H_\alpha 1 - u \leq 0 \quad \text{on } \partial U = (\partial U \setminus \partial D) \cup \partial D.$$

Therefore, we have the assertion

$$H_\alpha 1 \leq u \quad \text{on } \overline{U} = U \cup \partial U,$$

and hence

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) \leq \frac{\partial u}{\partial \mathbf{n}} \leq -K \quad \text{on } \partial D,$$

since $u|_{\partial D} = H_\alpha 1|_{\partial D} = 1$.

The proof of Lemma 9.16 is complete. □

In the following we shall use the notation

$$\|\varphi\|_\infty = \max_{x' \in \partial D} |\varphi(x')|$$

for a function $\varphi(x')$ defined on the boundary ∂D.

Now we can study the behavior of the operator norm $\| - \overline{L_N H_\alpha}^{-1}\|$ as $\alpha \to +\infty$:

Corollary 9.17. $\lim_{\alpha \to +\infty} \| - \overline{L_N H_\alpha}^{-1}\| = 0$.

Proof. By Lemma 9.16, it follows that the function

$$L_N H_\alpha 1(x') = \frac{\partial}{\partial \mathbf{n}} (H_\alpha 1)(x'), \quad x' \in \partial D,$$

diverges to $-\infty$ monotonically as $\alpha \to +\infty$. By Dini's theorem, this convergence is uniform in $x' \in \partial D$. Hence the function

$$\frac{1}{L_N H_\alpha 1(x')}$$

converges to zero uniformly in $x' \in \partial D$ as $\alpha \to +\infty$. This implies that

$$\begin{aligned}\left\| -\overline{L_N H_\alpha}^{-1} \right\| &= \left\| -\overline{L_N H_\alpha}^{-1} 1 \right\|_\infty \\ &\leq \left\| \frac{1}{L_N H_\alpha 1} \right\|_\infty \longrightarrow 0 \quad \text{as} \quad \alpha \to +\infty.\end{aligned}$$

Indeed, it suffices to note that

$$1 = \frac{-L_N H_\alpha 1(x')}{|L_N H_\alpha 1(x')|} \leq \left\| \frac{1}{L_N H_\alpha 1} \right\|_\infty (-L_N H_\alpha 1(x')) \quad \text{for all } x' \in \partial D.$$

The proof of Corollary 9.17 is complete. □

Step 7-2: **Proof of Assertion (9.30)**

In view of formula (9.25) and inequality (9.26), it suffices to prove that

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha^N f - f\|_\infty = 0, \quad f \in C^\infty(\overline{D}), \tag{9.36}$$

since the space $C^\infty(\overline{D})$ is dense in $C(\overline{D})$.

First, we remark that

$$\begin{aligned}\left\|\alpha G_\alpha^N f - f\right\|_\infty &= \left\|\alpha G_\alpha^0 f - \alpha H_\alpha\left(\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 f\right)\right) - f\right\|_\infty \\ &\le \left\|\alpha G_\alpha^0 f + H_\alpha\left(f|_{\partial D}\right) - f\right\|_\infty \\ &\quad + \left\|-\alpha H_\alpha\left(\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 f\right)\right) - H_\alpha\left(f|_{\partial D}\right)\right\|_\infty \\ &\le \left\|\alpha G_\alpha^0 f + H_\alpha\left(f|_{\partial D}\right) - f\right\|_\infty \\ &\quad + \left\|-\alpha\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 f\right) - f|_{\partial D}\right\|_\infty .\end{aligned}$$

Thus, in view of assertion (9.31) it suffices to show that

$$\lim_{\alpha\to+\infty}\left[-\alpha\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 f\right) - f|_{\partial D}\right] = 0 \quad \text{in } C(\partial D). \tag{9.37}$$

We take a constant β such that $0 < \beta < \alpha$, and write

$$f = G_\beta^0 g + H_\beta\varphi,$$

where (cf. formula (9.13)):

$$\begin{cases} g = (\beta - A)f \in C^\theta(\overline{D}), \\ \varphi = f|_{\partial D} \in C^{2+\theta}(\partial D). \end{cases}$$

Then, by using equations (9.6) (with $f := g$) and (9.9) we obtain that

$$\begin{aligned}G_\alpha^0 f &= G_\alpha^0 G_\beta^0 g + G_\alpha^0 H_\beta\varphi \\ &= \frac{1}{\alpha-\beta}\left(G_\beta^0 g - G_\alpha^0 g + H_\beta\varphi - H_\alpha\varphi\right).\end{aligned}$$

Hence we have the assertion

$$\begin{aligned}&\left\|-\alpha\overline{L_N H_\alpha}^{-1}\left(L_N G_\alpha^0 f\right) - f|_{\partial D}\right\|_\infty \\ &= \left\|\frac{\alpha}{\alpha-\beta}\left(-\overline{L_N H_\alpha}^{-1}\right)\left(L_N G_\beta^0 g - L_N G_\alpha^0 g + L_N H_\beta\varphi\right) + \frac{\alpha}{\alpha-\beta}\varphi - \varphi\right\|_\infty \\ &\le \frac{\alpha}{\alpha-\beta}\left\|-\overline{L_N H_\alpha}^{-1}\right\| \cdot \left\|L_N G_\beta^0 g + L_N H_\beta\varphi\right\|_\infty \\ &\quad + \frac{\alpha}{\alpha-\beta}\left\|-\overline{L_N H_\alpha}^{-1}\right\| \cdot \left\|L_N G_\alpha^0\right\| \cdot \|g\|_\infty + \frac{\beta}{\alpha-\beta}\|\varphi\|_\infty .\end{aligned}$$

By Corollary 9.17, it follows that the first term on the last inequality converges to zero as $\alpha \to +\infty$:

$$\frac{\alpha}{\alpha-\beta}\left\|-\overline{L_N H_\alpha}^{-1}\right\| \cdot \left\|L_N G_\beta^0 g + L_N H_\beta\varphi\right\|_\infty \longrightarrow 0 \quad \text{as } \alpha \to +\infty.$$

For the second term, by using formula (9.6) with $f := 1$ and the non-negativity of G_β^0 and $L_N G_\alpha^0$ we find that

$$\begin{aligned}\|L_N G_\alpha^0\| &= \|L_N G_\alpha^0 1\|_\infty \\ &= \|L_N G_\beta^0 1 - (\alpha-\beta) L_N G_\alpha^0 G_\beta^0 1\|_\infty \\ &\leq \|L_N G_\beta^0 1\|_\infty .\end{aligned}$$

Hence the second term also converges to zero as $\alpha \to +\infty$:

$$\frac{\alpha}{\alpha-\beta} \left\| -\overline{L_N H_\alpha}^{-1} \right\| \cdot \left\| L_N G_\alpha^0 \right\| \cdot \|g\|_\infty \longrightarrow 0 \quad \text{as } \alpha \to +\infty.$$

It is clear that the third term converges to zero as $\alpha \to +\infty$:

$$\frac{\beta}{\alpha-\beta} \|\varphi\|_\infty \longrightarrow 0 \quad \text{as } \alpha \to +\infty.$$

Therefore, we have proved assertion (9.37) and hence the desired assertion (9.36).

The proof of assertion (9.30) is complete.

Step 8: Summing up, we have proved that the operator $\mathfrak{A}_N$, defined by formula (9.20), satisfies conditions (a) through (d) in Theorem 2.16. Hence it follows from an application of the same theorem that the operator $\mathfrak{A}_N$ is the infinitesimal generator of some Feller semigroup on $\overline{D}$.

The proof of Theorem 9.14 is now complete. □

9.3 Proof of Theorem 1.4

We apply part (ii) of Theorem 2.16 to the operator $\mathfrak{A}$ defined by formula (1.5).

First, we simplify the boundary condition

$$Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \quad \text{on } \partial D.$$

Assume that the following conditions (A) and (B′) are satisfied:

(A) $\mu(x') \geq 0$ on ∂D.
(B′) $\gamma(x') \leq 0$ on ∂D and $\gamma(x') < 0$ on $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then it follows that

$$\mu(x') - \gamma(x') > 0 \quad \text{on } \partial D.$$

Thus we find that the boundary condition

$$\mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \quad \text{on } \partial D$$

is equivalent to the boundary condition

$$\left(\frac{\mu(x')}{\mu(x') - \gamma(x')} \right) \frac{\partial u}{\partial \mathbf{n}} + \left(\frac{\gamma(x')}{\mu(x') - \gamma(x')} \right) u = 0 \quad \text{on } \partial D.$$

However, if we let

$$\tilde{\mu}(x') = \frac{\mu(x')}{\mu(x') - \gamma(x')}, \quad \tilde{\gamma}(x') = \frac{\gamma(x')}{\mu(x') - \gamma(x')},$$

then we have the assertions

$$\tilde{\mu}(x') \frac{\partial u}{\partial \mathbf{n}} + \tilde{\gamma}(x') u = 0 \quad \text{on } \partial D$$

and

$$\begin{aligned} &0 \leq \tilde{\mu}(x') \leq 1 \quad \text{on } \partial D, \\ &\tilde{\gamma}(x') = \tilde{\mu}(x') - 1 \quad \text{on } \partial D. \end{aligned}$$

Namely, we may assume that the boundary condition L is of the form

$$Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + (\mu(x') - 1) u = 0 \quad \text{on } \partial D,$$

with

$$0 \leq \mu(x') \leq 1 \quad \text{on } \partial D.$$

Next we express the boundary condition L in terms of the Dirichlet and Neumann conditions.

It follows from an application of Lemmas 9.8 and 9.10 that

$$\overline{LG_\alpha^0} = \mu(x')\, \overline{L_N G_\alpha^0},$$

and that

$$\overline{LH_\alpha} = \mu(x')\, \overline{L_N H_\alpha} + \mu(x') - 1.$$

Hence, in view of definition (9.18) we obtain that

$$\begin{aligned} Lu &= \overline{LG_\alpha^0} \left((\alpha I - \overline{A})u\right) + \overline{LH_\alpha}(u|_{\partial D}) \\ &= \mu(x')\overline{L_N G_\alpha^0} \left((\alpha I - \overline{A})u\right) + \mu(x')\overline{L_N H_\alpha}(u|_{\partial D}) + (\mu(x') - 1)(u|_{\partial D}) \\ &= \mu(x') \left(\overline{L_N G_\alpha^0} \left((\alpha I - \overline{A})u\right) + \overline{L_N H_\alpha}(u|_{\partial D})\right) + (\mu(x') - 1)(u|_{\partial D}) \\ &= \mu(x') L_N u + (\mu(x') - 1)(u|_{\partial D}), \quad u \in \mathcal{D}(L). \end{aligned}$$

This proves the desired formula

$$L = \mu(x') L_N + \mu(x') - 1. \tag{9.38}$$

Therefore, the next theorem proves Theorem 1.4:

Theorem 9.18. *We define a linear operator*

$$\mathfrak{A} : C_0(\overline{D} \setminus M) \longrightarrow C_0(\overline{D} \setminus M)$$

as follows (cf. formula (9.20)).

(a) The domain $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$\mathcal{D}(\mathfrak{A}) = \Big\{ u \in C_0(\overline{D} \setminus M) : \overline{A}u \in C_0(\overline{D} \setminus M), \quad Lu = \mu(x')L_N u + (\mu(x') - 1)\,(u|_{\partial D}) = 0 \Big\}. \tag{9.39}$$

(b) $\mathfrak{A}u = \overline{A}u$, $u \in \mathcal{D}(\mathfrak{A})$.

Assume that the following condition (A′) is satisfied:

(A′) $0 \leq \mu(x') \leq 1$ on ∂D.

Then the operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D} \setminus M$, and the Green operator $G_\alpha = (\alpha I - \mathfrak{A})^{-1}$, $\alpha > 0$, is given by the following formula:

$$G_\alpha f = G_\alpha^N f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(L G_\alpha^N f \right) \right), \quad f \in C_0(\overline{D} \setminus M). \tag{9.40}$$

Here G_α^N is the Green operator for the Neumann condition L_N given by formula (9.21).

Proof. We apply part (ii) of Theorem 2.16 to the operator $\mathfrak{A}$. The proof is divided into six steps.

Step 1: First, we prove that:

> If condition (A′) is satisfied, then the operator $\overline{LH_\alpha}$ is the generator of some Feller semigroup on ∂D, for any *sufficiently large* $\alpha > 0$.

We introduce a linear operator

$$\mathcal{T}(\alpha) : B^{2-1/p,p}(\partial D) \longrightarrow B^{2-1/p,p}(\partial D)$$

as follows.

(a) The domain $\mathcal{D}(\mathcal{T}(\alpha))$ of $\mathcal{T}(\alpha)$ is the space

$$\mathcal{D}(\mathcal{T}(\alpha)) = \left\{ \varphi \in B^{2-1/p,p}(\partial D) : LH_\alpha \varphi \in B^{2-1/p,p}(\partial D) \right\}.$$

(b) $\mathcal{T}(\alpha)\varphi = LH_\alpha \varphi$, $\varphi \in \mathcal{D}(\mathcal{T}(\alpha))$.

Then, by arguing just as in the proof of Theorem 7.1 with $\gamma(x') := \mu(x') - 1$ on ∂D we obtain that

$$\begin{aligned} &\text{The operator } \mathcal{T}(\alpha) \text{ is } \textit{bijective} \text{ for any sufficiently large } \alpha > 0. \\ &\text{Furthermore, it maps the space } C^\infty(\partial D) \text{ onto itself.} \end{aligned} \tag{9.41}$$

Since we have the assertion

$$LH_\alpha = \mathcal{T}(\alpha) \quad \text{on } C^\infty(\partial D),$$

it follows from assertion (9.41) that the operator LH_α also maps $C^\infty(\partial D)$ onto itself, for any sufficiently large $\alpha > 0$. This implies that the range $\mathcal{R}(LH_\alpha)$ is a *dense* subset of $C(\partial D)$. Hence, by applying part (ii) of Theorem 9.12 we obtain that the operator $\overline{LH_\alpha}$ generates a Feller semigroup on ∂D, for any sufficiently large $\alpha > 0$.

Step 2: Next we prove that:

> The operator $\overline{LH_\beta}$ generates a Feller semigroup on ∂D,
> for *any* $\beta > 0$.

We take a positive constant α so large that the operator $\overline{LH_\alpha}$ generates a Feller semigroup on ∂D. We apply Corollary 2.19 with $K := \partial D$ to the operator $\overline{LH_\beta}$ for $\beta > 0$. By formula (9.16), it follows that the operator $\overline{LH_\beta}$ can be written as

$$\overline{LH_\beta} = \overline{LH_\alpha} + M_{\alpha\beta},$$

where $M_{\alpha\beta} = (\alpha - \beta)\overline{LG_\alpha^0}\, H_\beta$ is a bounded linear operator on $C(\partial D)$ into itself. Furthermore, assertion (9.15) implies that the operator $\overline{LH_\beta}$ satisfies condition (β') of Theorem 2.18. Therefore, it follows from an application of Corollary 2.19 that the operator $\overline{LH_\beta}$ also generates a Feller semigroup on ∂D.

Step 3: Now we prove that:

> If condition (A′) is satisfied, then the equation
> $\overline{LH_\alpha}\,\psi = \varphi$
> has a unique solution ψ in $\mathcal{D}\left(\overline{LH_\alpha}\right)$ for any $\varphi \in C(\partial D)$; hence the inverse $\overline{LH_\alpha}^{-1}$ of $\overline{LH_\alpha}$ can be defined on the whole space $C(\partial D)$.
> Furthermore, the operator $-\overline{LH_\alpha}^{-1}$ is non-negative and bounded on the space $C(\partial D)$. (9.42)

Since we have, by inequality (9.24),

$$LH_\alpha 1 = \mu(x')\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) + \mu(x') - 1 < 0 \quad \text{on } \partial D,$$

it follows that

$$k_\alpha = \min_{x' \in \partial D}(-LH_\alpha 1(x')) = -\max_{x' \in \partial D} LH_\alpha 1(x') > 0.$$

In view of Lemma 9.16, we find that the constants k_α are increasing in $\alpha > 0$:

$$\alpha \geq \beta > 0 \implies k_\alpha \geq k_\beta.$$

Furthermore, by using Corollary 2.17 with $K := \partial D$, $A := \overline{LH_\alpha}$ and $c := k_\alpha$ we obtain that the operator $\overline{LH_\alpha} + k_\alpha I$ is the infinitesimal generator of some

Feller semigroup on ∂D. Therefore, since $k_\alpha > 0$, it follows from an application of part (i) of Theorem 2.16 with $\mathfrak{A} := \overline{LH_\alpha} + k_\alpha I$ that the equation

$$-\overline{LH_\alpha}\,\psi = \left(k_\alpha I - (\overline{LH_\alpha} + k_\alpha I)\right)\psi = \varphi$$

has a unique solution $\psi \in \mathcal{D}\left(\overline{LH_\alpha}\right)$ for any $\varphi \in C(\partial D)$, and further that the operator $-\overline{LH_\alpha}^{-1} = \left(k_\alpha I - (\overline{LH_\alpha} + k_\alpha I)\right)^{-1}$ is non-negative and bounded on the space $C(\partial D)$ with norm

$$\left\|-\overline{LH_\alpha}^{-1}\right\| = \left\|\left(k_\alpha I - (\overline{LH_\alpha} + k_\alpha I)\right)^{-1}\right\| \leq \frac{1}{k_\alpha}.$$

Step 4: By assertion (9.42), we can define the operator G_α by formula (9.40) for all $\alpha > 0$. We prove that:

$$G_\alpha = (\alpha I - \mathfrak{A})^{-1}, \quad \alpha > 0. \tag{9.43}$$

By Lemma 9.6 and Theorem 9.14, it follows that we have, for all $f \in C_0(\overline{D} \setminus M)$,

$$G_\alpha f \in \mathcal{D}(\overline{A}),$$

and

$$\overline{A}(G_\alpha f) = \alpha G_\alpha f - f.$$

Furthermore, we obtain that the function $G_\alpha f$ satisfies the boundary condition

$$L(G_\alpha f) = LG_\alpha^N f - \overline{LH_\alpha}\left(\overline{LH_\alpha}^{-1}\left(LG_\alpha^N f\right)\right) = 0 \quad \text{on } \partial D. \tag{9.44}$$

However, we recall that (see formula (9.38))

$$Lu = \mu(x')L_N u + (\mu(x') - 1)\left(u|_{\partial D}\right), \quad u \in \mathcal{D}(L). \tag{9.45}$$

Hence it follows that the boundary condition (9.44) is equivalent to the following:

$$L(G_\alpha f) = \mu(x')L_N(G_\alpha f) + (\mu(x') - 1)\left(G_\alpha f|_{\partial D}\right) = 0 \quad \text{on } \partial D. \tag{9.46}$$

By condition (9.46), we have, for all $f \in C_0(\overline{D} \setminus M)$,

$$G_\alpha f = 0 \text{ on } M = \{x' \in \partial D : \mu(x') = 0\},$$

and so

$$\overline{A}(G_\alpha f) = \alpha G_\alpha f - f = 0 \quad \text{on } M.$$

Summing up, we have proved that

$$\begin{aligned} &f \in C_0(\overline{D} \setminus M) \\ &\Longrightarrow G_\alpha f \in \mathcal{D}(\mathfrak{A}) = \left\{u \in C_0(\overline{D} \setminus M) : \overline{A}u \in C_0(\overline{D} \setminus M),\, Lu = 0\right\}, \end{aligned}$$

and further that

$$(\alpha I - \mathfrak{A})G_\alpha f = f, \quad f \in C_0(\overline{D} \setminus M).$$

This proves that

$$(\alpha I - \mathfrak{A})G_\alpha = I \quad \text{on} \quad C_0(\overline{D} \setminus M).$$

Therefore, in order to prove formula (9.43), it suffices to show the injectivity of the operator $\alpha I - \mathfrak{A}$ for $\alpha > 0$.

Assume that, for $\alpha > 0$,

$$u \in \mathcal{D}(\mathfrak{A}) \text{ and } (\alpha I - \mathfrak{A})u = 0.$$

Then, by Corollary 9.7 it follows that the function u can be written as follows:

$$u = H_\alpha \left(u|_{\partial D} \right), \ u|_{\partial D} \in \mathcal{D} = \mathcal{D}\left(\overline{LH_\alpha}\right).$$

Thus we have the formula

$$\overline{LH_\alpha} \left(u|_{\partial D} \right) = Lu = 0.$$

In view of assertion (9.42), this implies that

$$u|_{\partial D} = 0,$$

so that

$$u = H_\alpha \left(u|_{\partial D} \right) = 0 \quad \text{in } D.$$

This proves the injectivity of $\alpha I - \mathfrak{A}$ for $\alpha > 0$.

Step 5: Now we prove the following three assertions:

(i) The operator G_α is non-negative on the space $C_0(\overline{D} \setminus M)$:

$$f \in C_0(\overline{D} \setminus M), \ f \geq 0 \quad \text{on } \overline{D} \setminus M \implies G_\alpha f \geq 0 \quad \text{on } \overline{D} \setminus M. \tag{9.47}$$

(ii) The operator G_α is bounded on the space $C_0(\overline{D} \setminus M)$ with norm

$$\|G_\alpha\| \leq \frac{1}{\alpha} \quad \text{for all } \alpha > 0. \tag{9.48}$$

(iii) The domain $\mathcal{D}(\mathfrak{A})$ is dense in the space $C_0(\overline{D} \setminus M)$.

Step 5-1: In order to prove assertion (i), we have only to show the non-negativity of the operator G_α on the space $C(\overline{D})$:

$$f \in C(\overline{D}), \ f \geq 0 \quad \text{on } \overline{D} \implies G_\alpha f \geq 0 \quad \text{on } \overline{D}. \tag{9.49}$$

Recall that the Dirichlet problem

$$\begin{cases} (\alpha - A)u = f & \text{in } D, \\ u = \varphi & \text{on } \partial D \end{cases} \tag{9.2}$$

is uniquely solvable. Hence it follows that

$$G_\alpha^N f = H_\alpha \left(G_\alpha^N f|_{\partial D} \right) + G_\alpha^0 f \quad \text{on } \overline{D}. \tag{9.50}$$

Indeed, the both sides satisfy the same equation $(\alpha - A)u = f$ in D and have the same boundary values $G_\alpha^N f$ on ∂D.

Thus, by applying the boundary operator L to the both sides of formula (9.50) we obtain that

$$LG_\alpha^N f = \overline{LH_\alpha} \left(G_\alpha^N f|_{\partial D} \right) + \overline{LG_\alpha^0} f.$$

Since the operators $-\overline{LH_\alpha}^{-1}$ and $\overline{LG_\alpha^0}$ are non-negative, it follows that

$$\begin{aligned} & f \geq 0 \quad \text{on } \overline{D} \\ & \Longrightarrow \\ & \left(-\overline{LH_\alpha}^{-1} \right) \left(LG_\alpha^N f \right) = -G_\alpha^N f|_{\partial D} + \left(-\overline{LH_\alpha}^{-1} \right) \left(\overline{LG_\alpha^0} f \right) \\ & \geq -G_\alpha^N f|_{\partial D} \text{ on } \partial D. \end{aligned}$$

Therefore, by the non-negativity of H_α and G_α^0 we find from formulas (9.40) and (9.50) that

$$\begin{aligned} G_\alpha f &= G_\alpha^N f + H_\alpha \left(-\overline{LH_\alpha}^{-1} \left(LG_\alpha^N f \right) \right) \\ &\geq G_\alpha^N f + H_\alpha \left(-G_\alpha^N f|_{\partial D} \right) \\ &= G_\alpha^0 f \\ &\geq 0 \quad \text{on } \overline{D}. \end{aligned}$$

This proves the desired assertion (9.49) and hence assertion (9.47).

Step 5-2: Next we prove assertion (ii). To do this, it suffices to show the boundedness of the operator G_α on the space $C(\overline{D})$:

$$G_\alpha 1 \leq \frac{1}{\alpha} \quad \text{on } \overline{D}, \tag{9.51}$$

since G_α is non-negative on the space $C(\overline{D})$.

We remark (cf. formula (9.45)) that

$$\begin{aligned} LG_\alpha^N f &= \mu(x')\, L_N G_\alpha^N f + (\mu(x') - 1) \left(G_\alpha^N f|_{\partial D} \right) \\ &= (\mu(x') - 1) \left(G_\alpha^N f|_{\partial D} \right), \end{aligned}$$

so that

$$\begin{aligned} G_\alpha f &= G_\alpha^N f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(LG_\alpha^N f \right) \right) \\ &= G_\alpha^N f + H_\alpha \left(-\overline{LH_\alpha}^{-1} \left((\mu(x') - 1) G_\alpha^N f|_{\partial D} \right) \right). \end{aligned}$$

Hence, by using this formula with $f := 1$ we obtain that

$$G_\alpha 1 = G_\alpha^N 1 - H_\alpha \left(-\overline{LH_\alpha}^{-1} \left((1-\mu(x'))G_\alpha^N 1|_{\partial D} \right) \right).$$

However, we have, by inequality (9.27),

$$0 \leq G_\alpha^N 1 \leq \frac{1}{\alpha} \quad \text{on } \overline{D},$$

and also

$$H_\alpha \left(-\overline{LH_\alpha}^{-1} \left((1-\mu(x'))G_\alpha^N 1|_{\partial D} \right) \right) \geq 0 \quad \text{on } \overline{D},$$

since the operators H_α and $-\overline{LH_\alpha}^{-1}$ are non-negative and since $1-\mu(x') \geq 0$ on ∂D.

Therefore, we obtain that

$$0 \leq G_\alpha 1 \leq G_\alpha^N 1 \leq \frac{1}{\alpha} \quad \text{on } \overline{D}.$$

This proves the desired assertion (9.51) and hence assertion (9.48).

Step 5-3: Finally, we prove assertion (iii). In view of formula (9.43), it suffices to show that

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha f - f\|_\infty = 0, \quad f \in C_0(\overline{D} \setminus M) \cap C^\infty(\overline{D}), \tag{9.52}$$

since the space $C_0(\overline{D} \setminus M) \cap C^\infty(\overline{D})$ is dense in $C_0(\overline{D} \setminus M)$.

We remark that

$$\begin{aligned} \alpha G_\alpha f - f &= \alpha G_\alpha^N f - f - \alpha H_\alpha \left(\overline{LH_\alpha}^{-1} \left(LG_\alpha^N f \right) \right) \\ &= \left(\alpha G_\alpha^N f - f \right) + H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\alpha(1-\mu(x'))G_\alpha^N f|_{\partial D} \right) \right). \end{aligned} \tag{9.53}$$

We estimate the last two terms of formula (9.53) as follows:

(1) By assertion (9.36), it follows that the first term of formula (9.53) tends to zero as $\alpha \to +\infty$:

$$\lim_{\alpha \to +\infty} \left\| \alpha G_\alpha^N f - f \right\|_\infty = 0. \tag{9.54}$$

(2) To estimate the second term of formula (9.53), we remark that

$$\begin{aligned} &H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\alpha(1-\mu(x'))G_\alpha^N f|_{\partial D} \right) \right) \\ &= H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x'))f|_{\partial D} \right) \right) \\ &\quad + H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x'))(\alpha G_\alpha^N f - f)|_{\partial D} \right) \right). \end{aligned} \tag{9.55}$$

However, it follows that the second term in the right-hand side of formula (9.55) tends to zero as $\alpha \to +\infty$. Indeed, we have, by assertion (9.54),

$$\begin{aligned}&\left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) \left(\alpha G_\alpha^N f - f \right) |_{\partial D} \right) \right) \right\|_\infty \\ &\leq \left\| -\overline{LH_\alpha}^{-1} \right\| \cdot \left\| (1-\mu(x')) \left(\alpha G_\alpha^N f - f \right) |_{\partial D} \right\|_\infty \\ &\leq \frac{1}{k_\alpha} \left\| (1-\mu(x')) \left(\alpha G_\alpha^N f - f \right) |_{\partial D} \right\|_\infty \\ &\leq \frac{1}{k_1} \left\| \alpha G_\alpha^N f - f \right\|_\infty \longrightarrow 0. \end{aligned} \tag{9.56}$$

Here we have used the following facts (cf. the proof of assertion (9.42)):

$$\left\| -\overline{LH_\alpha}^{-1} \right\| \leq \frac{1}{k_\alpha}, \quad \alpha > 0.$$

$$k_1 = \min_{x' \in \partial D} (-LH_1 1)(x') \leq k_\alpha = \min_{x' \in \partial D} (-LH_\alpha 1)(x') \quad \text{for all } \alpha \geq 1.$$

Thus we are reduced to the study of the first term of the right-hand side of formula (9.55)

$$H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) f|_{\partial D} \right) \right).$$

Now, for any given $\varepsilon > 0$, we can find a function $h(x')$ in $C^\infty(\partial D)$ such that

$$\begin{cases} h = 0 \quad \text{near } M = \{x' \in \partial D : \mu(x') = 0\}, \\ \|(1-\mu(x')) f|_{\partial D} - h\|_\infty < \varepsilon. \end{cases}$$

Then we have, for all $\alpha \geq 1$,

$$\begin{aligned}&\left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) f|_{\partial D} \right) \right) - H_\alpha \left(\overline{LH_\alpha}^{-1} h \right) \right\|_\infty \\ &\leq \left\| -\overline{LH_\alpha}^{-1} \right\| \cdot \|(1-\mu(x')) f|_{\partial D} - h\|_\infty \\ &\leq \frac{\varepsilon}{k_\alpha} \\ &\leq \frac{\varepsilon}{k_1}. \end{aligned} \tag{9.57}$$

Furthermore, we can find a function $\theta(x')$ in $C_0^\infty(\partial D)$ such that

$$\begin{cases} \theta(x') = 1 & \text{near } M, \\ (1-\theta(x')) h(x') = h(x') & \text{on } \partial D. \end{cases}$$

Then we have the assertion

$$\begin{aligned} h(x') &= (1-\theta(x'))\, h(x') \\ &= (-LH_\alpha 1(x')) \left(\frac{1-\theta(x')}{-LH_\alpha 1(x')} \right) h(x') \\ &\leq \left\| \frac{1-\theta}{-LH_\alpha 1} \right\|_\infty \cdot \|h\|_\infty \, (-LH_\alpha 1(x')). \end{aligned}$$

Since the operator $-\overline{LH_\alpha}^{-1}$ is non-negative on the space $C(\partial D)$, it follows that

$$-\overline{LH_\alpha}^{-1} h \leq \left\| \frac{1-\theta}{-LH_\alpha 1} \right\|_\infty \cdot \|h\|_\infty \quad \text{on } \partial D,$$

so that

$$\left\| H_\alpha \left(\overline{LH_\alpha}^{-1} h \right) \right\| \leq \left\| -\overline{LH_\alpha}^{-1} h \right\|_\infty \leq \left\| \frac{1-\theta}{-LH_\alpha 1} \right\|_\infty \cdot \|h\|_\infty . \tag{9.58}$$

However, there exists a positive constant δ_0 such that

$$0 \leq \frac{1-\theta(x')}{\mu(x')} \leq \delta_0 \quad \text{for all } x' \in \partial D.$$

Thus it follows that

$$\begin{aligned} \frac{1-\theta(x')}{-LH_\alpha 1(x')} &= \frac{1-\theta(x')}{\mu(x') \left(-\frac{\partial}{\partial \mathbf{n}} \left(H_\alpha 1(x') \right) \right) + (1-\mu(x'))} \\ &\leq \left(\frac{1-\theta(x')}{\mu(x')} \right) \frac{1}{\left(-\frac{\partial}{\partial \mathbf{n}} \left(H_\alpha 1(x') \right) \right)} \\ &\leq \delta_0 \, \frac{1}{\min_{x' \in \partial D} \left(-\frac{\partial}{\partial \mathbf{n}} \left(H_\alpha 1(x') \right) \right)}, \end{aligned}$$

and hence from Lemma 9.16 that

$$\lim_{\alpha \to +\infty} \left\| \frac{1-\theta}{-LH_\alpha 1} \right\|_\infty = 0. \tag{9.59}$$

Summing up, we obtain from inequalities (9.57) and (9.58) and assertion (9.59) that

$$\begin{aligned} &\limsup_{\alpha \to +\infty} \left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) f|_{\partial D} \right) \right) \right\|_\infty \\ &\leq \limsup_{\alpha \to +\infty} \left[\left\| H_\alpha \left(\overline{LH_\alpha}^{-1} h \right) \right\| \right. \\ &\qquad \left. + \left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) f|_{\partial D} \right) \right) - H_\alpha \left(\overline{LH_\alpha}^{-1} h \right) \right\|_\infty \right] \\ &\leq \lim_{\alpha \to +\infty} \left\| \frac{1-\theta}{-LH_\alpha 1} \right\|_\infty \cdot \|h\|_\infty + \frac{\varepsilon}{k_1} \\ &\leq \frac{\varepsilon}{k_1}. \end{aligned}$$

Since ε is arbitrary, this proves that the first term of the right-hand side of formula (9.55) tends to zero as $\alpha \to +\infty$:

$$\lim_{\alpha \to +\infty} \left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left((1-\mu(x')) f|_{\partial D} \right) \right) \right\|_\infty = 0. \tag{9.60}$$

By assertions (9.56) and (9.60), we obtain that the last term of formula (9.53) also tends to zero:

$$\lim_{\alpha \to +\infty} \left\| H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\alpha(1 - \mu(x'))G_\alpha^N f|_{\partial D} \right) \right\|_\infty = 0. \tag{9.61}$$

Therefore, the desired assertion (9.52) follows by combining assertions (9.54) and (9.61).

The proof of assertion (iii) is complete.

Step 6: Summing up, we have proved that the operator $\mathfrak{A}$, defined by formula (9.39), satisfies conditions (a) through (d) in Theorem 2.16. Hence, in view of assertion (8.2), it follows from an application of part (ii) of the same theorem that the operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D} \setminus M$.

The proof of Theorem 9.18 and hence that of Theorem 1.4 is now complete. □

9.4 Proof of Part (ii) of Theorem 1.3

We apply Theorem 2.2 to the operator $\mathfrak{A}$.

In the proof of Theorem 9.18, we have proved that the domain $\mathcal{D}(\mathfrak{A})$ is *dense* in the space $C_0(\overline{D} \setminus M)$. Furthermore, part (i) of Theorem 1.3 verifies condition (2.1). Therefore, it follows from an application of Theorem 2.2 that:

> The semigroup T_t can be extended to a semigroup T_z which is *analytic* in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, \, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$.

This (together with Theorem 1.4) proves part (ii) of Theorem 1.3. □

10

Application to Semilinear Initial-Boundary Value Problems

This chapter is devoted to the semigroup approach to a class of initial-boundary value problems for *semilinear* parabolic differential equations. We prove Theorem 1.5 by using the theory of fractional powers of analytic semigroups (Theorems 10.1 and 10.2). To do this, we verify that all the conditions of Theorem 2.8 are satisfied. Our semigroup approach here can be traced back to the pioneering work of Fujita–Kato [FK]. For detailed studies of semilinear parabolic equations, the reader is referred to Friedman [Fr1], Henry [He] and also [Ta4].

10.1 Local Existence and Uniqueness Theorems

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$A = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a second-order, *elliptic* differential operator with real smooth coefficients on $\overline{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$ and $1 \leq i, j \leq N$.
(2) There exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x, \xi) \in \overline{D} \times \mathbf{R}^N.$$

(3) $c(x) \leq 0$ on $\overline{D}$.

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As an application of Theorem 1.2, we consider the following *semilinear* initial-boundary value problem: Given functions $f(x,t,u,\xi)$ and $u_0(x)$ defined in $D \times [0,T) \times \mathbf{R} \times \mathbf{R}^N$ and in D, respectively, find a function $u(x,t)$ in $D \times [0,T)$ such that

$$\begin{cases} \left(\frac{\partial}{\partial t} - A\right) u(x,t) = f(x,t,u,\operatorname{grad} u) & \text{in } D \times (0,T), \\ Lu(x',t) := \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 & \text{on } \partial D \times [0,T), \\ u(x,0) = u_0(x) & \text{in } D. \end{cases} \tag{1.7}$$

Here:

(4) $\mu \in C^\infty(\partial D)$ and $\mu(x') \geq 0$ on ∂D.
(5) $\gamma \in C^\infty(\partial D)$ and $\gamma(x') \leq 0$ on ∂D.
(6) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D (see Figure 1.1).

Recall that the operator A_p is a unbounded linear operator from $L^p(D)$ into itself given by the following formulas:

(a) The domain of definition $\mathcal{D}(A_p)$ of A_p is the space

$$\mathcal{D}(A_p) = \left\{ u \in H^{2,p}(D) = W^{2,p}(D) : Lu(x') = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \right\}.$$

(b) $A_p u = Au$, $u \in \mathcal{D}(A_p)$.

By using the operator A_p, we can formulate problem (1.7) in terms of the abstract *Cauchy problem* in the Banach space $L^p(D)$ as follows:

$$\begin{cases} \frac{du}{dt} = A_p u(t) + F(t, u(t)), & 0 < t < T, \\ u|_{t=0} = u_0. \end{cases} \tag{1.8}$$

Here $u(t) = u(\cdot, t)$ and $F(t,u(t)) = f(\cdot, t, u(t), \operatorname{grad} u(t))$ are functions defined on the interval $[0,T)$, taking values in the space $L^p(D)$.

First, we consider the case $N < p < \infty$:

Theorem 10.1. *Let $N < p < \infty$, and assume that conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ on ∂D.
(B) $\gamma(x') < 0$ on $M = \{x' \in D : \mu(x') = 0\}$.

If the nonlinear term $f(x,t,u,\xi)$ is a locally Lipschitz continuous function with respect to all its variables $(x,t,u,\xi) \in D \times [0,T) \times \mathbf{R} \times \mathbf{R}^N$ with the possible exception of the x variables, then, for every function u_0 of $\mathcal{D}(A_p)$, problem (1.8) has a unique local solution $u \in C([0,T_1]; L^p(D)) \cap C^1((0,T_1); L^p(D))$ where $T_1 = T_1(p, u_0)$ is a positive constant.

Here $C\left([0,T];\, L^p(D)\right)$ denotes the space of continuous functions on the closed interval $[0,T]$ taking values in $L^p(D)$, and $C^1\left((0,T);\, L^p(D)\right)$ denotes the space of continuously differentiable functions on the open interval $(0,T)$ taking values in $L^p(D)$, respectively.

In the case $1 < p < N$, the domain $\mathcal{D}(A_p)$ is large compared with the case $N < p < \infty$. Hence we must impose some growth conditions on the nonlinear term $f(x,t,u,\xi)$:

Theorem 10.2. *Let $N/2 < p < N$, and assume that conditions (A) and (B) are satisfied. Furthermore, we assume that there exist a non-negative continuous function $\rho(t,r)$ on $\mathbf{R} \times \mathbf{R}$ and a constant $1 \leq \gamma < N/(N-p)$ such that the following four conditions are satisfied:*

(a) $|f(x,t,u,\xi)| \leq \rho(t,|u|)(1+|\xi|^\gamma)$.
(b) $|f(x,t,u,\xi) - f(x,s,u,\xi)| \leq \rho(t,|u|)\,(1+|\xi|^\gamma)\,|t-s|$.
(c) $|f(x,t,u,\xi) - f(x,t,u,\eta)| \leq \rho(t,|u|)\left(1+|\xi|^{\gamma-1}+|\eta|^{\gamma-1}\right)|\xi-\eta|$.
(d) $|f(x,t,u,\xi) - f(x,t,v,\xi)| \leq \rho(t,|u|+|v|)\,(1+|\xi|^\gamma)\,|u-v|$.

Then, for every function u_0 of $\mathcal{D}(A_p)$, problem (1.8) has a unique local solution $u \in C\left([0,T_2];\, L^p(D)\right) \cap C^1\left((0,T_2);\, L^p(D)\right)$ where $T_2 = T_2(p,u_0)$ is a positive constant.

Theorems 10.1 and 10.2 prove Theorem 1.5, and are a generalization of Pazy [Pa, Section 8.4, Theorems 4.4 and 4.5] to the *degenerate* case.

10.2 Fractional Powers and Imbedding Theorems

First, we study the imbedding properties of the domains of the fractional powers $(-A_p)^\alpha$ $(0 < \alpha < 1)$ into Sobolev spaces of L^p type. By virtue of Theorem 2.8, this allows us to solve, by successive approximations, problem (1.8), proving Theorems 10.1 and 10.2.

By Theorem 7.1, we may assume that the operator A_p satisfies condition (2.21) in Subsection 2.1.2 (see Figure 7.1):

(1) The resolvent set of A_p contains the region Σ as in Figure 10.1:
(2) There exists a positive constant M such that the resolvent $(A_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\|(A_p - \lambda I)^{-1}\right\| \leq \frac{M}{(1+|\lambda|)} \quad \text{for all } \lambda \in \Sigma. \tag{10.1}$$

By using estimate (10.1), we can define the fractional powers $(-A_p)^\alpha$ for $0 < \alpha < 1$ on the space $L^p(D)$ as follows (cf. formula (2.23)):

$$(-A_p)^{-\alpha} = -\frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha}(A_p - sI)^{-1}\, ds,$$

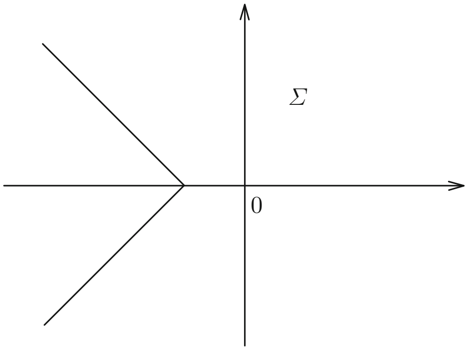

Fig. 10.1.

and

$$(-A_p)^\alpha = \text{ the inverse of } (-A_p)^{-\alpha}.$$

We recall that $(-A_p)^\alpha$ is a closed operator with domain $\mathcal{D}\left((-A_p)^\alpha\right) \supset \mathcal{D}(A_p)$.

In this section we study the imbedding characteristics of $\mathcal{D}((-A_p)^\alpha)$, which will make these spaces so useful in the study of semilinear parabolic differential equations.

We let

$$X_\alpha = \text{the space } \mathcal{D}\left((-A_p)^\alpha\right) \text{ endowed with the graph norm } \|\cdot\|_\alpha \text{ of } (-A_p)^\alpha.$$

Here

$$\|u\|_\alpha = \left(\|u\|_p^2 + \|(-A_p)^\alpha u\|_p^2\right)^{1/2}, \quad u \in \mathcal{D}\left((-A_p)^\alpha\right).$$

Then we have the following three assertions:

(1) The space X_α is a Banach space.
(2) The graph norm $\|u\|_\alpha$ is equivalent to the norm $\|(-A_p)^\alpha u\|_p$.
(3) If $0 < \alpha < \beta < 1$, then we have $X_\beta \subset X_\alpha$ with continuous injection.

Furthermore, since the domain $\mathcal{D}(A_p)$ is contained in $W^{2,p}(D)$, we can obtain the following imbedding properties of the spaces X_α into Sobolev spaces (cf. [He, Theorem 1.6.1]):

Theorem 10.3. *Let $1 < p < \infty$. Then we have the following continuous injections:*

(i) $X_\alpha \subset W^{1,q}(D)$ *if* $\frac{1}{2} < \alpha < 1$, $\frac{1}{p} - \frac{2\alpha-1}{N} < \frac{1}{q} \leq \frac{1}{p}$, $1 < p < N$.
(ii) $X_\alpha \subset C^\nu(\overline{D})$ *if* $\frac{N}{2p} < \alpha < 1$, $0 \leq \nu < 2\alpha - \frac{N}{p}$, $p \neq N$.

10.3 Proof of Theorems 10.1 and 10.2

This section is devoted to the proof of our local existence and uniqueness theorems for problem (1.8) (Theorems 10.1 and 10.2). To do this, we verify that all the conditions of Theorem 2.8 are satisfied.

10.3.1 Proof of Theorem 10.1

We verify that all the conditions of Theorem 2.8 are satisfied; then Theorem 10.1 follows from an application of the same theorem.

Since $p > N$, we can choose a positive constant α such that

$$\frac{1}{2}\left(\frac{N}{p}+1\right) < \alpha < 1,$$

so that

$$1 < 2\alpha - \frac{N}{p}.$$

Then, by part (ii) of Theorem 10.3 with $\nu := 1$, we have

$$X_\alpha \subset C^1(\overline{D}) \quad \text{and} \quad X_\alpha \subset W^{1,p}(D), \tag{10.2}$$

with continuous injections. Thus we find that the function

$$F(t,u) := f(x,t,u(x),\operatorname{grad} u(x))$$

is well defined on $[0,T] \times X_\alpha$. Furthermore, since the function $f(x,t,u,\xi)$ is locally Lipschitz continuous, in view of assertion (10.2) it follows that we have, for all t, $s \in [0,t_0]$ and for all u, $v \in X_\alpha$ with $\|u-u_0\|_\alpha \leq R$, $\|v-u_0\|_\alpha \leq R$

$$\begin{aligned}
\|F(t,u)-F(s,v)\|_p &\leq \|F(t,u)-F(t,v)\|_p + \|F(t,v)-F(s,v)\|_p \\
&\leq C\left(\|u-v\|_p + \sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(u-v)\right\|_p + |t-s|\right) \\
&\leq C\left(\|u-v\|_{1,p} + |t-s|\right) \\
&\leq CC'\left(\|u-v\|_\alpha + |t-s|\right).
\end{aligned} \tag{10.3}$$

Here $C = C(t_0,R)$ is a (local) Lipschitz positive constant for the function f, and C' is an imbedding (positive) constant for the imbedding: $X_\alpha \subset W^{1,p}(D)$.

By inequality (10.3), we obtain that the function $F(t,u)$ is locally Lipschitz continuous in t and u.

The proof of Theorem 10.1 is complete. □

10.3.2 Proof of Theorem 10.2

The proof is similar to that of Theorem 10.1; we verify that all the conditions of Theorem 2.8 are satisfied.

Since $N/2 < p < N$ and $1 \leq \gamma < N/(N-p)$, we can choose a positive constant α such that

$$\max\left(\frac{N}{2p}, \frac{1}{2} + \frac{N}{2p}\left(\frac{\gamma-1}{\gamma}\right)\right) < \alpha < 1, \tag{10.4}$$

so that

$$0 < 2\alpha - \frac{N}{p} \quad \text{and} \quad \frac{1}{p} - \frac{2\alpha-1}{N} < \frac{1}{p\gamma} \leq \frac{1}{p}.$$

Then, by Theorem 10.3 with $\nu := 0$ and $q := p\gamma$, we have

$$X_\alpha \subset L^\infty(D) \quad \text{and} \quad X_\alpha \subset W^{1,p\gamma}(D), \tag{10.5}$$

with continuous injections.

We let

$$F(t,u) := f(x,t,u(x), \operatorname{grad} u(x)), \quad t \in [0,T],\, u \in X_\alpha.$$

Then we have, by condition (a),

$$\begin{aligned}\|F(t,u)\|_p^p &\leq 2^{p-1}\rho(t, \|u\|_\infty)^p \int_D (1 + |\operatorname{grad} u|^{p\gamma})dx \\ &\leq 2^{p-1}\rho(t, \|u\|_\infty)^p \left(|D| + \|u\|_{1,p\gamma}^{p\gamma}\right),\end{aligned}$$

where $|D|$ denotes the volume of the domain D. By assertion (10.5), it follows that the function $F(t,u)$ is well defined on $[0,T] \times X_\alpha$ for all α satisfying condition (10.4).

Step 1: First, we verify the local Lipschitz continuity of $F(t,u)$ with respect to the variable t.

By condition (b), it follows that

$$\begin{aligned}&\|F(t,u) - F(s,u)\|_p^p \\ &= \int_D |f(x,t,u(x), \operatorname{grad} u(x)) - f(x,s,u(x), \operatorname{grad} u(x))|^p dx \\ &\leq 2^{p-1}\rho(t, \|u\|_\infty)^p |t-s|^p \int_D (1 + |\operatorname{grad} u|^{p\gamma})dx \\ &\leq 2^{p-1}\rho(t, \|u\|_\infty)^p \left(|D| + \|u\|_{1,p\gamma}^{p\gamma}\right) |t-s|^p.\end{aligned}$$

In view of assertion (10.5), this proves that

$$\|F(t,u) - F(s,u)\|_p \leq C_1(\|u\|_\alpha)\, |t-s|, \tag{10.6}$$

where $C_1(\|u\|_\alpha)$ is a positive constant depending on the norm $\|u\|_\alpha$.

Step 2: Secondly, we verify the local Lipschitz continuity of $F(t,u)$ with respect to the variable u.

To do this, we remark that

$$\begin{aligned}
&\|F(t,u) - F(t,v)\|_p^p \\
&= \int_D |f(x,t,u(x),\operatorname{grad} u(x)) - f(x,t,v(x),\operatorname{grad} v(x))|^p dx \\
&\leq 2^{p-1} \int_D |f(x,t,u(x),\operatorname{grad} u(x)) - f(x,t,u(x),\operatorname{grad} v(x))|^p dx \\
&\quad + 2^{p-1} \int_D |f(x,t,u(x),\operatorname{grad} v(x)) - f(x,t,v(x),\operatorname{grad} v(x))|^p dx. \qquad (10.7)
\end{aligned}$$

We estimate the two terms on the right-hand side of inequality (10.7):

Step 2-1: By condition (c), it follows that

$$\begin{aligned}
&\int_D |f(x,t,u(x),\operatorname{grad} u(x)) - f(x,t,u(x),\operatorname{grad} v(x))|^p \, dx \\
&\leq 3^{p-1} \rho(t, \|u\|_\infty)^p \\
&\quad \times \int_D \left(1 + |\operatorname{grad} u|^{p(\gamma-1)} + |\operatorname{grad} v|^{p(\gamma-1)}\right) |\operatorname{grad}(u-v)|^p \, dx. \qquad (10.8)
\end{aligned}$$

However, by Hölder's inequality it follows that

$$\begin{aligned}
\int_D |\operatorname{grad}(u-v)|^p \, dx &\leq \left(\int_D 1 dx\right)^{(\gamma-1)/\gamma} \left(\int_D |\operatorname{grad}(u-v)|^{p\gamma} dx\right)^{1/\gamma} \\
&\leq |D|^{(\gamma-1)/\gamma} \|u-v\|_{1,p\gamma}^p, \qquad (10.9)
\end{aligned}$$

$$\begin{aligned}
\int_D |\operatorname{grad} u|^{p(\gamma-1)} |\operatorname{grad}(u-v)|^p \, dx &\leq \left(\int_D |\operatorname{grad} u|^{p\gamma}\right)^{(\gamma-1)/\gamma} \\
&\quad \times \left(\int_D |\operatorname{grad}(u-v)|^{p\gamma} dx\right)^{1/\gamma} \\
&\leq \|u\|_{1,p\gamma}^{p(\gamma-1)} \|u-v\|_{1,p\gamma}^p \qquad (10.10)
\end{aligned}$$

$$\int_D |\operatorname{grad} v|^{p(\gamma-1)} \, |\operatorname{grad}(u-v)|^p dx \leq \|v\|_{1,p\gamma}^{p(\gamma-1)} \, \|u-v\|_{1,p\gamma}^p. \qquad (10.11)$$

Thus, by carrying these three inequalities (10.9), (10.10) and (10.11) into inequality (10.8) we obtain that

$$\begin{aligned}
&\int_D |f(x,t,u(x),\operatorname{grad} u(x)) - f(x,t,u(x),\operatorname{grad} v(x))|^p \, dx \\
&\leq 3^{p-1} \rho(t, \|u\|_\infty)^p \left(|D|^{(\gamma-1)/\gamma} + \|u\|_{1,p\gamma}^{p(\gamma-1)} + \|v\|_{1,p\gamma}^{p(\gamma-1)}\right) \\
&\quad \times \|u-v\|_{1,p\gamma}^p. \qquad (10.12)
\end{aligned}$$

Step 2-2: By condition (d), it follows that

$$\begin{aligned}
&\int_D |f(x,t,u(x),\operatorname{grad} v(x)) - f(x,t,v(x),\operatorname{grad} v(x))|^p \, dx \\
&\le 2^{p-1}\rho(t, \|u\|_\infty + \|v\|_\infty)^p \int_D (1 + |\operatorname{grad} v|^{p\gamma})\, |u-v|^p dx \\
&\le 2^{p-1}\rho(t, \|u\|_\infty + \|v\|_\infty)^p \, \|u-v\|_\infty^p \int_D (1 + |\operatorname{grad} v|^{p\gamma})\, dx \\
&\le 2^{p-1}\rho(t, \|u\|_\infty + \|v\|_\infty)^p \left(|D| + \|v\|_{1,p\gamma}^{p\gamma}\right) \|u-v\|_\infty^p. \qquad (10.13)
\end{aligned}$$

Therefore, by combining inequalities (10.7), (10.12) and (10.13) we obtain that

$$\begin{aligned}
&\|F(t,u) - F(t,v)\|_p^p \\
&\le 6^{p-1}\rho(t, \|u\|_\infty)^p \left(|D|^{(\gamma-1)/\gamma} + \|u\|_{1,p\gamma}^{p(\gamma-1)} + \|v\|_{1,p\gamma}^{p(\gamma-1)}\right) \|u-v\|_{1,p\gamma}^p \\
&\quad + 4^{p-1}\rho(t, \|u\|_\infty + \|v\|_\infty)^p \left(|D| + \|v\|_{1,p\gamma}^{p\gamma}\right) \|u-v\|_\infty^p.
\end{aligned}$$

In view of assertion (10.5), this proves that

$$\|F(t,u) - F(t,v)\|_p \le C_2(\|u\|_\alpha, \|v\|_\alpha)\, \|u-v\|_\alpha, \qquad (10.14)$$

where $C_2(\|u\|_\alpha, \|v\|_\alpha)$ is a positive constant depending on the norms $\|u\|_\alpha$ and $\|v\|_\alpha$.

Summing up, we find from inequalities (10.6) and (10.14) that the function $F(t,u)$ is locally Lipschitz continuous in t and u.

The proof of Theorem 10.2 is now complete. □

11

Concluding Remarks

This book is devoted to a careful and accessible exposition of the functional analytic approach to the problem of construction of Markov processes with Ventcel' boundary conditions in probability theory. More precisely, we prove that there exists a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the state space $\overline{D} \setminus M$ until it "dies" at the time when it reaches the set M where the particle is definitely absorbed (see Figure 11.1). Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of pseudo-differential operators which may be considered as a modern theory of the classical potential theory.

More generally, it is known (see [BCP], [SU], [Ta2], [We]) that the infinitesimal generator $\mathfrak{W}$ of a Feller semigroup $\{T_t\}_{t \geq 0}$ is described analytically by a Waldenfels operator W and a Ventcel' boundary condition L, which we formulate precisely.

Let W be a second-order, *elliptic* integro-differential operator with real coefficients such that

$$\begin{aligned} Wu(x) &= Au(x) + S_r u(x) \\ &:= \left(\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \right) \\ &\quad + \int_D s(x,y) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right] dy. \end{aligned}$$

Here:

(1) $a^{ij} \in C^\infty(\overline{D})$, $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$ and $1 \leq i,j \leq N$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbf{R}^N.$$

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Lecture Notes in Mathematics 1499, DOI 10.1007/978-3-642-01677-6_11,

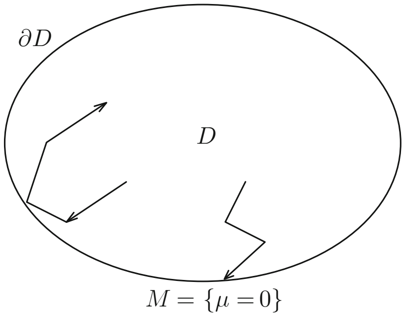

Fig. 11.1.

(2) $b^i \in C^\infty(\overline{D})$ for all $1 \leq i \leq N$.
(3) $c \in C^\infty(\overline{D})$ and $c(x) \leq 0$ in D.
(4) The integral kernel $s(x, y)$ is the distribution kernel of a properly supported, pseudo-differential operator $S \in L^{2-\kappa}_{1,0}(\mathbf{R}^N)$, $\kappa > 0$, which has the *transmission property* with respect to ∂D (see [Bt]), and $s(x, y) \geq 0$ off the diagonal $\{(x, x) : x \in \mathbf{R}^N\}$ in $\mathbf{R}^N \times \mathbf{R}^N$. The measure dy is the Lebesgue measure on $\mathbf{R}^N$.

The operator W is called a second-order, *Waldenfels operator* (cf. [Wa]). The differential operator A is called a diffusion operator which describes analytically a strong Markov process with continuous paths (diffusion process) in the interior D. The operator S_r is called a second-order Lévy operator which is supposed to correspond to the jump phenomenon in the interior D. More precisely, a Markovian particle moves by jumps to a random point, chosen with kernel $s(x, y)$, in the interior D. Therefore, the Waldenfels operator $W = A + S_r$ is supposed to correspond to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space D (see Figure 11.2).

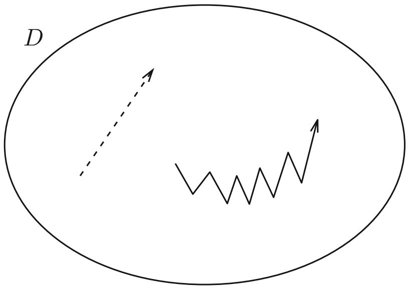

Fig. 11.2.

Let L be a second-order boundary condition such that, in local coordinates $(x_1, x_2, \ldots, x_{N-1})$,

$$
\begin{aligned}
Lu(x') &= Qu(x') + \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') W u(x') + \Gamma u(x') \\
&:= \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x') + \gamma(x') u(x') \right) \\
&\quad + \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') W u(x') \\
&\quad + \left(\int_{\partial D} r(x', y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy' \right. \\
&\quad \left. + \int_D t(x', y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy \right).
\end{aligned}
$$

Here:

(1) The operator Q is a second-order, degenerate elliptic differential operator on the boundary ∂D with non-positive principal symbol. In other words, the α^{ij} are the components of a smooth symmetric contravariant tensor of type $\binom{2}{0}$ on ∂D satisfying the condition

$$
\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \xi_i \xi_j \geq 0 \quad \text{for all } x' \in \partial D \text{ and } \xi' = \textstyle\sum_{j=1}^{N-1} \xi_j dx_j \in T^*_{x'}(\partial D).
$$

Here $T^*_{x'}(\partial D)$ is the cotangent space of ∂D at x'.

(2) $Q1 = \gamma \in C^\infty(\partial D)$ and $\gamma(x') \leq 0$ on ∂D.

(3) $\mu \in C^\infty(\partial D)$ and $\mu(x') \geq 0$ on ∂D.

(4) $\delta \in C^\infty(\partial D)$ and $\delta(x') \geq 0$ on ∂D.

(5) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D.

(6) The integral kernel $r(x', y')$ is the distribution kernel of a pseudo-differential operator $R \in L^{2-\kappa_1}_{1,0}(\partial D)$, $\kappa_1 > 0$, and $r(x', y') \geq 0$ off the diagonal $\Delta_{\partial D} = \{(x', x') : x' \in \partial D\}$ in $\partial D \times \partial D$. The density dy' is a strictly positive density on ∂D.

(7) The integral kernel $t(x, y)$ is the distribution kernel of a properly supported, pseudo-differential operator $T \in L^{2-\kappa_2}_{1,0}(\mathbf{R}^N)$, $\kappa_2 > 0$, which has the *transmission property* with respect to the boundary ∂D (see [Bt]), and $t(x, y) \geq 0$ off the diagonal $\{(x, x) : x \in \mathbf{R}^N\}$ in $\mathbf{R}^N \times \mathbf{R}^N$.

The boundary condition L is called a second-order *Ventcel' boundary condition* (cf. [We]). The six terms of L

$$
\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x'),
$$

$$\gamma(x')u(x'), \quad \mu(x')\frac{\partial u}{\partial \mathbf{n}}(x'), \quad \delta(x')Wu(x'),$$

$$\int_{\partial D} r(x',y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy',$$

$$\int_{D} t(x',y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy$$

are supposed to correspond to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon, the sticking (viscosity) phenomenon and the jump phenomenon on the boundary and the inward jump phenomenon from the boundary, respectively (see Figures 11.3 through 11.5).

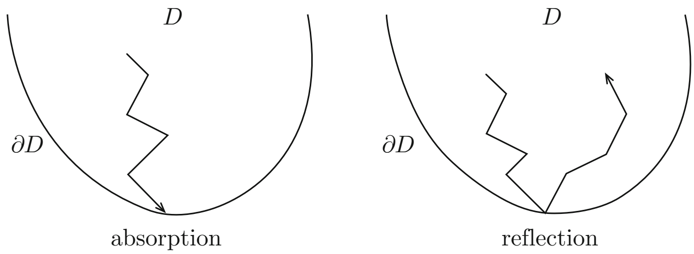

Fig. 11.3.

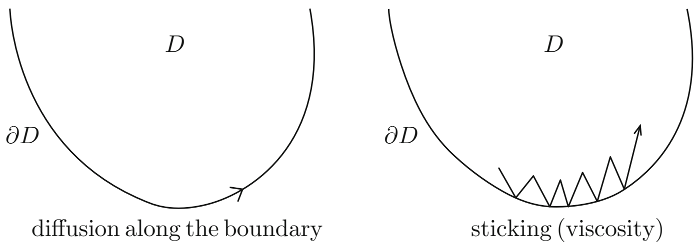

Fig. 11.4.

Finally, we give an overview for general results on generation theorems for Feller semigroups proved mainly by the author using the theory of pseudo-differential operators ([Ho1], [Se1], [Se2]) and the theory of singular integral operators ([CZ]):

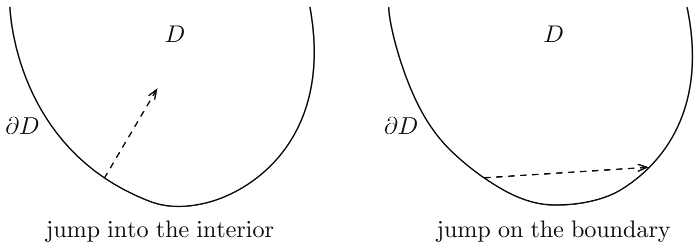

Fig. 11.5.

diffusion operator A	Lévy operator S_r	Ventcel' condition L	using the theory of	proved by
smooth case	null	second-order case	pseudo-differential operators	[Ta2]
smooth case	general case	general case	pseudo-differential operators	[Ta3]
smooth case	Hölder continuous case	degenerate case	pseudo-differential operators	[Ta5]
discon-tinuous case	general case	Dirichlet case	singular integral operators	[Ta6]
discon-tinuous case	null	first-order case	singular integral operators	[Ta7]

It should be emphasized that the Calderón–Zygmund theory of singular integral operators with non-smooth kernels provides a powerful tool to deal with smoothness of solutions of elliptic boundary value problems, with minimal assumptions of regularity on the coefficients. The theory of singular integrals continues to be one of the most influential works in modern history of analysis, and is a very refined mathematical tool whose full power is yet to be exploited (see [St2]).

A

The Maximum Principle

In this appendix, we formulate various maximum principles for second-order, elliptic differential operators such as the weak maximum principle (Theorem A.1) and the Hopf boundary point lemma (Lemma A.3) which play an important role in Chapter 9.

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with boundary ∂D, and let A be a second-order, elliptic differential operator with real coefficients such that

$$A = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial}{\partial x_i} + c(x).$$

Here:

(1) $a^{ij} \in C(\overline{D})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$, $1 \leq i, j \leq N$, and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{D} \times \mathbf{R}^N.$$

(2) $b^i \in C(\overline{D})$ for all $1 \leq i \leq N$.
(3) $c \in C(\overline{D})$ and $c(x) \leq 0$ in D.

First, we have the following weak maximum principle:

Theorem A.1 (the weak maximum principle). *Assume that a function $u \in C(\overline{D}) \cap C^2(\overline{D})$ satisfies either the condition*

$$Au(x) \geq 0 \quad \textit{and} \quad c(x) < 0 \quad \textit{in } D$$

or the condition

$$Au(x) > 0 \quad \textit{and} \quad c(x) \leq 0 \quad \textit{in } D.$$

Then the function $u(x)$ may take its positive maximum *only on the boundary ∂D.*

As an application of the weak maximum principle, we can obtain a pointwise estimate for solutions of the inhomogeneous equation $Au = f$ in D:

Theorem A.2. *Assume that*

$$c(x) < 0 \quad \text{on } \overline{D} = D \cup \partial D.$$

Then we have, for all $u \in C(\overline{D}) \cap C^2(D)$,

$$\max_{x \in \overline{D}} |u(x)| \leq \max \left\{ \left(\frac{1}{\min_{x \in \overline{D}}(-c(x))} \right) \sup_{x \in D} |Au(x)|, \ \max_{x' \in \partial D} |u(x')| \right\}.$$

Now we assume that D is a *domain of class* C^2, that is, each point of the boundary ∂D has a neighborhood in which ∂D is the graph of a C^2 function of $N-1$ of the variables $x_1, x_2, \ldots, x_N$ (see Figure A.1).

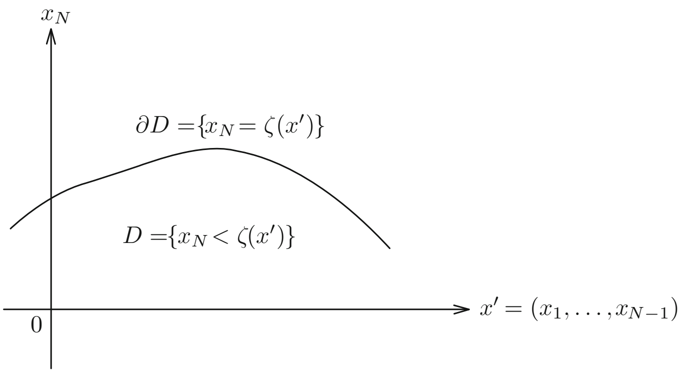

Fig. A.1.

We consider a function $u \in C(\overline{D}) \cap C^2(D)$ which satisfies the condition

$$Au(x) \geq 0 \quad \text{in } D,$$

and study the interior normal derivative $(\partial u)/(\partial \mathbf{n})$ at a boundary point where the function $u(x)$ takes its non-negative maximum (see Figure A.2).

The Hopf boundary point lemma reads as follows:

Lemma A.3 (the Hopf boundary point lemma). *Let D be a domain of class C^2. Assume that a function $u \in C(\overline{D}) \cap C^2(D)$ satisfies the condition*

$$Au(x) \geq 0 \quad \text{in } D,$$

and that there exists a point $x_0' \in \partial D$ such that

$$u(x_0') = \max_{x \in \overline{D}} u(x) \geq 0,$$
$$u(x) < u(x_0') \quad \text{for all } x \in D.$$

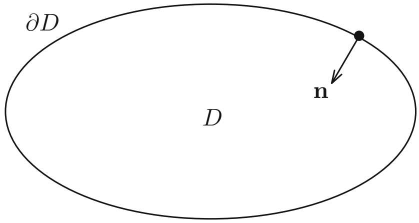

Fig. A.2.

Then the interior normal derivative $\frac{\partial u}{\partial \mathbf{n}}(x_0')$ *at* x_0'*, if it exists, satisfies the condition (see Figure A.3)*

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0.$$

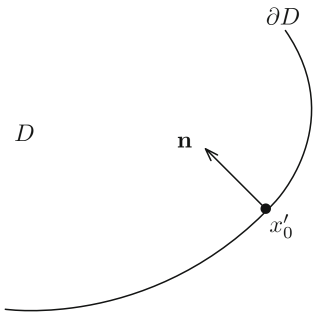

Fig. A.3.

For a proof of Theorems A.1, A.2 and Lemma A.3 and a general study of maximum principles, the reader might refer to [PW, Chapter 2] and [Ta2, Chapter 7] (see also [Ta5, Appendix C]).

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Edited by J.-M. Morel, F. Takens, B. Teissier, P.K. Maini

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